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ON THE ERRORS OF SPHERICAL HARMONIC DEVELOPMENTS OF GRAVITY AT THE SURFACE OF THE EARTH

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The Ohio State University
Research Foundation
Columbus, Ohio 43212

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The spherical harmonic expansion of the external gravity field of the earth is divergent at the surface of the earth. From the practical point of view a truncated series can be used if the errors are small. The errors are of two types: the truncation error and the downward continuation error. In this report we investigate the latter, which is caused by the disturbing topography above the point where the series is developed. This error is studied for potentials,

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del(Delta(g))/del(r)

gravity anomalies and for the vertical gradient of gravity. It is shown that the usual relation between the spherical harmonics of T and $\frac{\partial \Delta g}{\partial r}$ is also valid for their errors, while the relation between T_n and $\Delta(g_n)$ does not hold for the downward continuation errors. The relative errors of Δg are found to be more serious than those for T.

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Some global RMS errors are estimated based on the degree variances of Tscherning and Rapp (1974). Furthermore, formulae are developed for a numerical integration of the errors over an approximately known topography. Finally, these formulae are tested for 1654 $5° \times 5°$ mean elevations. In the earth model "a level ellipsoid with topography of constant density" these computations gave the RMS error 0.13 m for global undulations in an expansion to degree 16. The gravity anomaly errors were generally within ± 5 mgal except at the edges of the continents and for rough areas inside the continents, where larger errors might be expected.

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Foreword

This report was prepared by Lars Sjöberg, Research Associate, Department of Geodetic Science, The Ohio State University, under Air Force Contract No. F19628-76-C-0010, The Ohio State University Research Foundation Project No. 710335. The contract covering this research is administered by the Air Force Geophysics Laboratory, L. G. Hanscom Air Force Base, Massachusetts, with Mr. Bela Szabo, Contract Monitor.

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Table of Contents

1. Introduction

From potential theory it is well known that a series of spherical harmonics of the gravity field is convergent outside the "minimum sphere" enveloping all masses. In many studies it has been shown that for certain earth models the series may be extended to a sphere entirely within the surface of matter or the radius of convergence might be located somewhere between these extremes. For references, see Moritz (1961), Molodensky et al. (1962, pp. 118-120), Pick (1965), Morrison (1969), Hotine (1969, pp. 172-173) and Levallois (1972).

Moritz (ibid.) showed that if the earth were a homogeneous oblate spheroid (a level ellipsoid), then the series for V would converge at the surface. Levallois studied the condition for extension of the series down to the surface of a homogeneous, approximately spherical body. For such a body he found the following condition for convergence at the surface:

$$k_n < (0.132)^n$$

where

$$k_n^2 = \frac{\sum_{m=0}^{n} (\bar{C}_{nm}^2 + \bar{S}_{nm}^2)}{2n+1}$$

$\bar{C}_{nm}$, $\bar{S}_{nm}$ = fully normalized spherical harmonics

For the earth we have, according to Kaula's rule:

$$k_n \approx 10^{-5}/n^2 \tag{1.1}$$

Thus the condition above is not satisfied in this case. Some other models and the corresponding radii of convergence of the series are given by Morrison (ibid).

However, in all these studies the models are either homogeneous or bodies of revolution. Already a small disturbing body with mass centre located outside a homogeneous sphere makes the harmonic series divergent at the surface of the sphere (see the previous references of Moritz, Molodensky et.al., Pick and the example 3.1 below). Hence, because of the irregular mass distributions of the actual earth the series of spherical harmonics must be considered divergent at the surface of the earth (Moritz, 1961).

These theoretical aspects do not imply that the analytic continuation of gravimetric quantities down to the surface of the earth is meaningless. Moritz (1969) has paid attention to this fact and states that asymptotic series, which are mathematically speaking divergent, are frequently used in mathematical physics. "Such series can be used if the first terms decrease rapidly enough for their sum to provide a good estimation to the function to be calculated; it will not matter practically if the neglected higher terms will start increasing again. The practical use of divergent series needs, however, to be justified. An arbitrarily accurate approximation can be obtained only with convergent series; with divergent asymptotic series, this error cannot be reduced below a certain limit. It must be investigated whether this limit is small enough so as to be in keeping with the desired accuracy."

The main object of this report is to estimate the downward continuation error of the representation of the gravity field by a series of spherical harmonics at the surface of the earth. We start with a definition of the problem.

2. Definition of the Problem

In the volume external to a sphere enclosing all mass of the earth[1] (the minimum sphere, or Brillouin sphere) the gravity potential of the earth is harmonic and can be expanded into a series of spherical harmonics:

$$V = \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} V_n \tag{2.1}$$

where

$$V_n = \frac{GM}{R} \sum_{m=0}^{n} (\overline{C}_{nm} \cos m\lambda + \overline{S}_{nm} \sin m\lambda)\, \overline{P}_{nm}(\cos\theta)$$

G = Newton's constant of gravitation
M = mass of the earth
R = radius of the "minimum sphere"
(r, θ, λ) = spherical coordinates
$\overline{C}_{nm}$, $\overline{S}_{nm}$ = fully normalized spherical harmonic coefficients
$\overline{P}_{nm}(\cos\theta)$ = associated Legendre function

In many cases the potential V is substituted by the disturbing potential:

$$T = V - U$$

[1] The atmosphere of the earth is not considered

where U is the normal potential (usually consisting of the harmonics $U_0 = V_0$, U_{20} and U_{40}). In this case the series expansion in formula (2.1) starts at n = 2 and the coefficients for T_{20} and T_{40} are now considered as corrections to the normal potential.

Inserting the series expansion of T into the spherical approximation of the boundary condition of physical geodesy (see Heiskanen and Moritz, 1967, p. 88):

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2T}{r} \tag{2.2}$$

the gravity anomalies (Δg) are obtained in a series of the potential harmonics:

$$\Delta g = \sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^{n+2} \Delta g_n \tag{2.3}$$

where

$$\Delta g_n = \frac{GM}{R^2} (n-1) \sum_{m=0}^{n} (\overline{C}_{nm} \cos m\lambda + \overline{S}_{nm} \sin m\lambda)\, \overline{P}_{nm}(\cos\theta)$$

From (2.2) it is easily shown that $r\Delta g$ is harmonic in the same domain as T, i.e. outside the surface of the earth (if the influence of the earth's atmosphere is neglected).

The coefficients $\overline{C}_{nm}$ and $\overline{S}_{nm}$ of the series (2.1) and (2.2) have been determined to various degrees (N) from terrestrial gravity observations, satellite observations and from combinations thereof. From these coefficients the external gravity field can be determined by a truncated series:

$$\hat{V} = \sum_{n=0}^{N} \left(\frac{R}{r}\right)^{n+1} V_n \tag{2.4}$$

These truncated series are theoretically correct outside the "minimum sphere" (R = 6384.403 km for a = 6378.140 km, see Appendix). Now the question arises whether the series $\hat{V}$ and $\hat{\Delta g}$ can be analytically continued down to the surface of the earth. More precisely: What is the error of such a representation? Is it possible to find simple correction terms?

3. A Simple Model

Before going deeper into the problem of estimating the errors of using spherical harmonic series at the surface of the earth, we would like to present a simple, illustrative model.

A disturbing point mass m is located outside a homogeneous sphere M with radius r_0. The distance to m from the center of M is R. See Figure 3.1. At a point P of distance ℓ from m the disturbing potential is:

$$T = \frac{\mu}{\ell} = \frac{\mu}{(R^2 + r^2 - 2rR\cos\psi)^{\frac{1}{2}}}$$

where

μ = Gm
G = Newton's constant of gravitation
ψ = the angle between the radius vectors $\bar{r}$ and $\bar{R}$
r = distance from the center of M to P

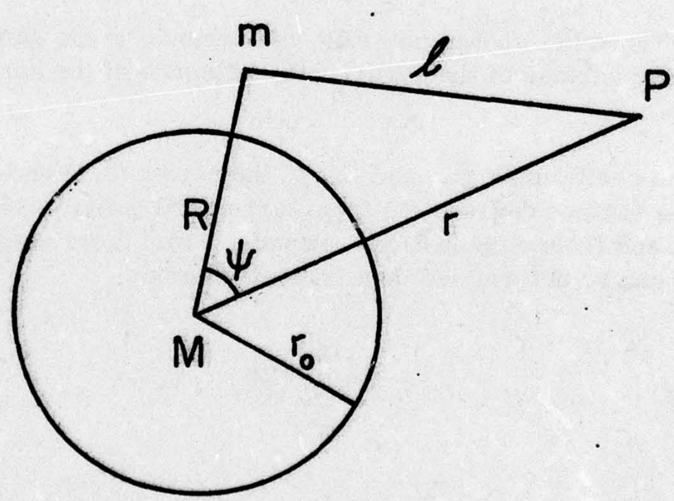

Figure 3.1. A homogeneous sphere with an exterior disturbing point mass m.

At a point outside a sphere with the same center as M and radius R the closed formula for T can be expanded in the following series of Legendre's polynomials:

$$T_e = \frac{\mu}{R} \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} P_n(\cos\psi), \qquad r > R \tag{3.1}$$

Applying this formula at the surface of M we obtain:

$$\hat{T} = \frac{\mu}{R} \sum_{n=0}^{N} \left(\frac{R}{r_0}\right)^{n+1} P_n(\cos\psi) \tag{3.1'}$$

This series is divergent for $N = \infty$, because $R > r_0$. The correct value for the disturbing potential at M is given by:

$$T_i = \frac{\mu}{(r_0^2 + R^2 - 2r_0 R\cos\psi)^{\frac{1}{2}}} = \frac{\mu}{r_0} \sum_{n=0}^{\infty} \left(\frac{r_0}{R}\right)^{n+1} P_n(\cos\psi) \tag{3.2}$$

Thus we obtain the following error of $\hat{T}$:

$$\epsilon_T(N) = \hat{T} - T_i = \delta T(N) + e_T(N)$$

where $\delta T(N)$ is the error of analytic continuation and $e_T(N)$ is the truncation error. The truncation error is caused by the truncation of $\hat{T}$ at degree N:

$$e_T(N) = -\frac{\mu}{r_0} \sum_{n=N+1}^{\infty} \left(\frac{r_0}{R}\right)^{n+1} P_n(\cos\psi)$$

The downward continuation error is defined by:

$$\delta T(N) = \sum_{n=0}^{N} \delta T_n$$

where δT_n is the error caused by the improper downward continuation of $(T_e)_n$ to the surface of the main sphere. We have:

$$\delta T_n = (T_e)_n - (T_i)_n = c_n (T_i)_n$$

where

$$(T_i)_n = \frac{\mu}{r_0} \left(\frac{r_0}{R}\right)^{n+1} P_n(\cos\psi)$$

and

$$c_n = \left(\frac{R}{r_0}\right)^{2n+1} - 1$$

The coefficient c_n is the relative error of $(T_i)_n$. Inserting $R = r_0 + h$ we obtain:

$$c_n = \left(1 + \frac{h}{r_0}\right)^{2n+1} - 1 \approx (2n+1)\,\frac{h}{r_0}$$

In Figure 3.2 the true value of the geoidal heights (T_i/γ) and some estimates for downward continuation are shown. In Figure 3.3 the relative error c_n is illustrated. Finally, in Figure 3.4 the relative error

$$\delta T(N) / \sum_{n=0}^{N} (T_i)_n$$

is given. The figures show that the relative errors are increasing with the elevation (h) of the disturbing mass and with the degree (n respective N).

Now we proceed to study the errors of the gravity anomalies. In the external case the radial derivative of T is obtained from (3.1):

$$\text{(3.3)} \qquad \left(\frac{\partial T}{\partial r}\right)_e = -\frac{\mu}{R^2} \sum_{n=0}^{\infty} (n+1) \left(\frac{R}{r}\right)^{n+2} P_n(\cos\psi) \qquad r > R$$

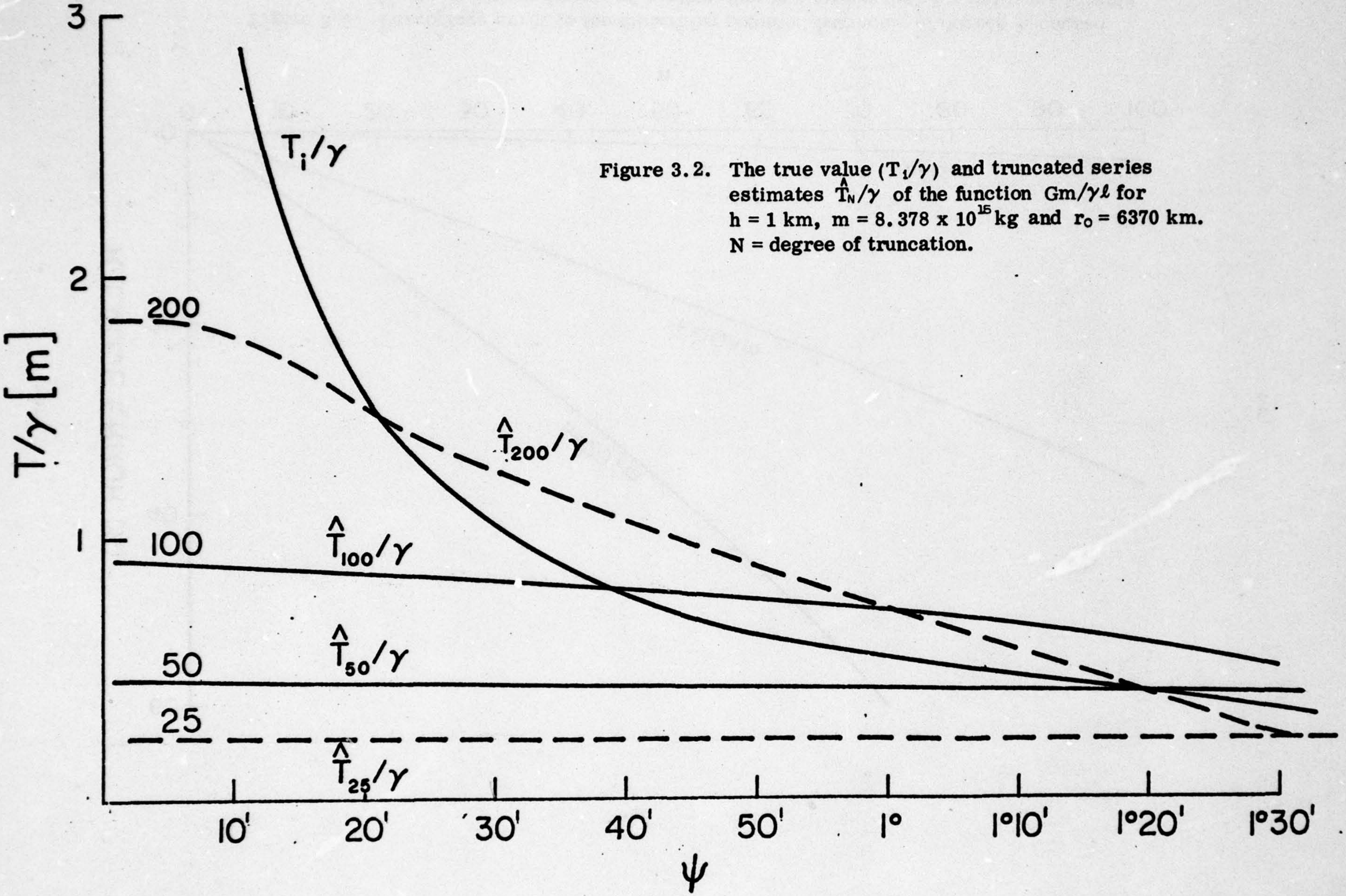

Figure 3.2. The true value (T_i/γ) and truncated series estimates $\hat{T}_N/\gamma$ of the function $Gm/\gamma \ell$ for $h = 1$ km, $m = 8.378 \times 10^{15}$ kg and $r_0 = 6370$ km. N = degree of truncation.

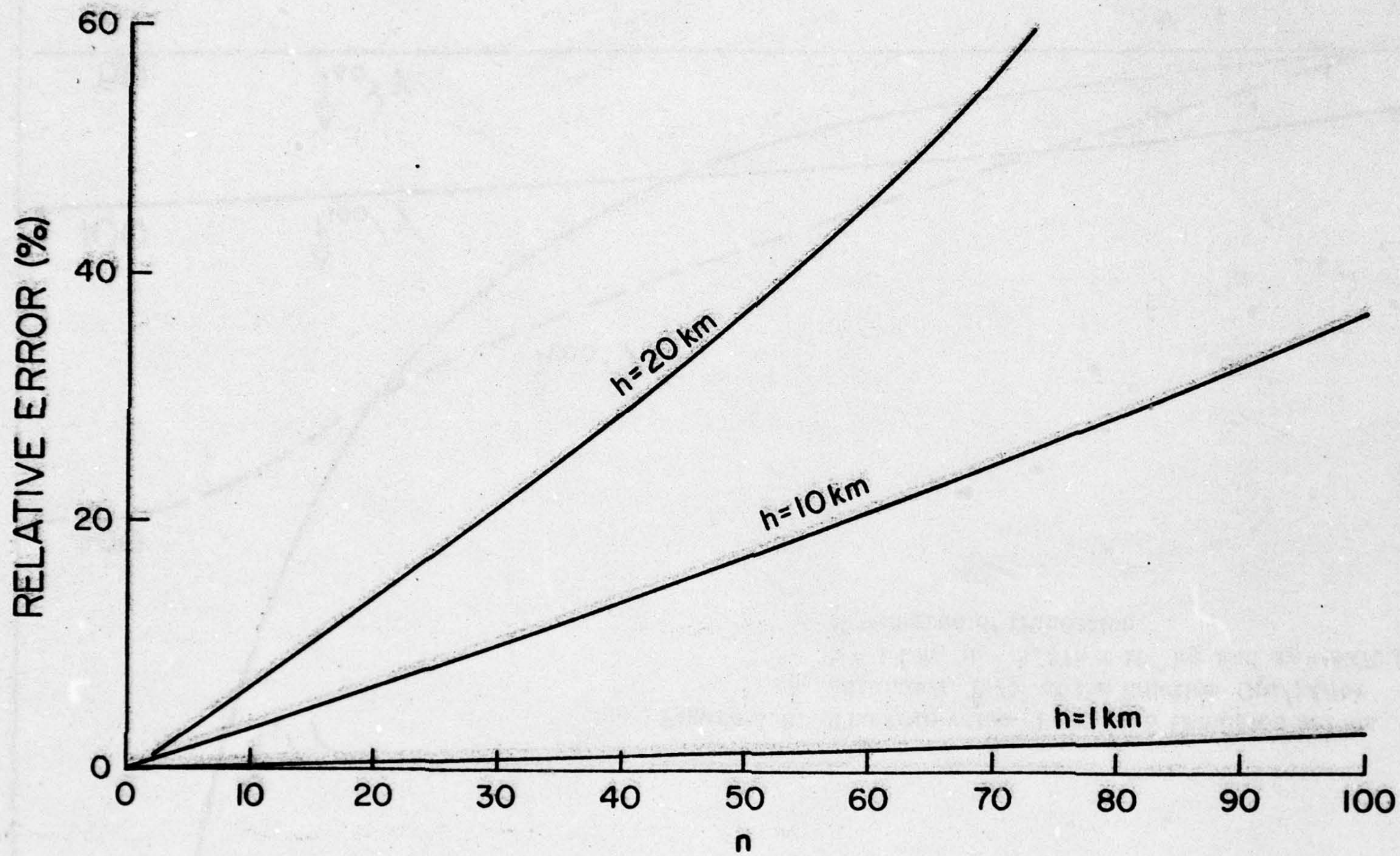

Figure 3.3. Percentage error in the disturbing potential harmonic of degree n caused by the improper downward continuation to a sphere with a point mass located at a height h above that sphere (r_o = 6370 km, ψ = 3').

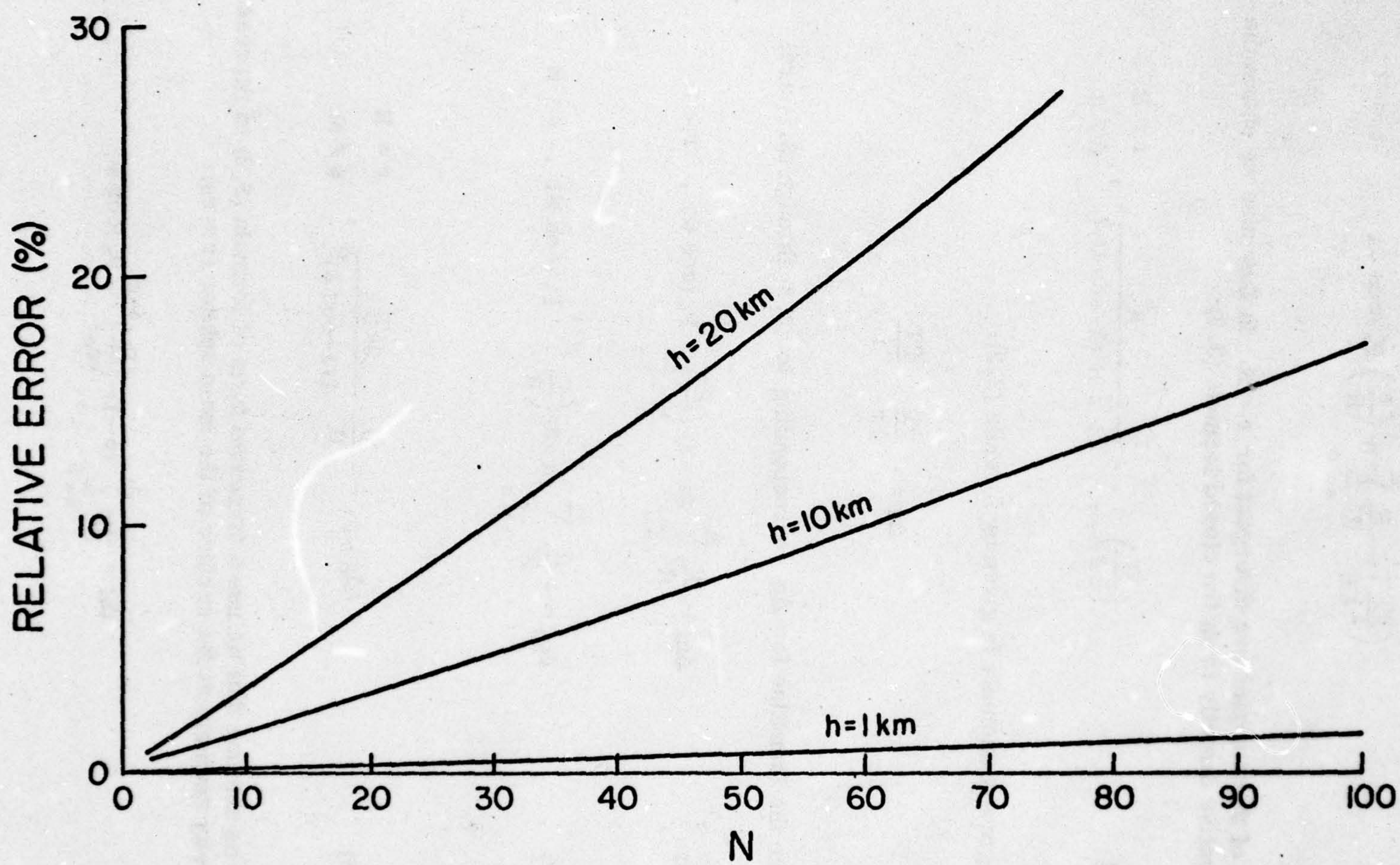

Figure 3.4. Percentage error in the disturbing potential harmonic expansion to degree N caused by the improper downward continuation to a sphere with a point mass located at a height h above that sphere (r_0 = 6370 km, ψ = 3').

For $r < R$ we have from (3.2):

$$\left(\frac{\partial T}{\partial r}\right)_i = \frac{\mu}{R^2}\sum_{n=0}^{\infty} n \left(\frac{r}{R}\right)^{n-1} P_n(\cos\psi) \qquad r < R \tag{3.4}$$

Both of these series are divergent for $r = R$. In this case we obtain the derivative directly from the closed formula (3.2):

$$\left(\frac{\partial T}{\partial r}\right)_{r=R} = - \frac{\mu}{2\sqrt{2}\, R^2 (1-\cos\psi)^{\frac{1}{2}}}\,, \qquad \begin{matrix} r = R \\ \psi \neq 0 \end{matrix} \tag{3.5}$$

The gravity anomaly is given by formula (2.2):

$$\Delta g = - \frac{\partial T}{\partial r} - \frac{2T}{r}$$

Hence the formulae for Δg corresponding to (3.3) through (3.5) are:

$$\Delta g_e = \frac{\mu}{R^2}\sum_{n=0}^{\infty} (n-1)\left(\frac{R}{r}\right)^{n+2} P_n(\cos\psi)\,, \quad r > R \tag{3.6}$$

$$\Delta g_i = -\frac{\mu}{R^2}\sum_{n=0}^{\infty} (n+2)\left(\frac{r}{R}\right)^{n-1} P_n(\cos\psi)\,, \quad r < R \tag{3.7}$$

and

$$(\Delta g)_{r=R} = -\frac{\mu}{R^2}\,\frac{3\sqrt{2}}{4(1-\cos\psi)^{\frac{1}{2}}}\,, \qquad \begin{matrix} r = R \\ \psi \neq 0 \end{matrix} \tag{3.8}$$

Let us assume that we use a truncated form of formula (3.6) to represent the gravity anomaly at the surface of the main sphere ($r = r_o$):

$$\hat{\Delta g} = \frac{\mu}{R^2}\sum_{n=0}^{N} (n-1)\left(\frac{R}{r_o}\right)^{n+2} P_n(\cos\psi)$$

where N is the degree of truncation. The true anomaly at this level is given by (3.7). Thus the total error of $\hat{\Delta g}$ becomes:

$$\epsilon_{\Delta g}(N) = \hat{\Delta g} - \Delta g_1 = \delta\,\Delta g\,(N) + e_{\Delta g}(N)$$

where $\delta\,\Delta g\,(N)$ is the error of the analytic continuation and $e_{\Delta g}(N)$ is the truncation error. We obtain:

$$e_{\Delta g}(N) = -\sum_{n=N+1}^{\infty} (\Delta g_1)_n$$

and

$$\delta\,\Delta g\,(N) = \sum_{n=0}^{N} \delta\,\Delta g_n$$

where

$$\delta\,\Delta g_n = d_n\,(\Delta g_1)_n$$

$$(\Delta g_1)_n = -\frac{\mu}{R^2}\,(n+2)\left(\frac{r_0}{R}\right)^{n-1} P_n\,(\cos\psi)$$

$$d_n = -\left[1 + \frac{n-1}{n+2}\left(\frac{R}{r_0}\right)^{2n+1}\right]$$

The coefficients c_n and $|d_n|$ are the relative errors of the n th harmonic of the anomalous potential and the gravity anomaly, respectively. c_n approaches zero for $R \rightarrow r_0$. This is not the case for d_n. The series Δg_e is not convergent for $r_0 = R$. The reader should notice that the usual relation between the harmonics of the potential and the gravity anomaly is not valid in this case (see Heiskanen and Moritz, 1967, p. 97):

$$\delta\,\Delta g_n \neq \frac{n-1}{r}\,\delta T_n$$

The influence of the disturbing masses on the anomalies is more pronounced than for the potentials. One reason for this is that while the disturbing potential is dependent only on the distance to the disturbing masses, the contribution to the gravity anomalies from masses located above the point of computation usually have opposite sign to that obtained in the downward continuation procedure.

Finally, we derive the errors of the vertical gradient of the gravity anomalies. We have:

$$\frac{\partial \Delta g_e}{\partial r} = - \frac{\mu}{R^3} \sum_n (n-1)(n+2) \left(\frac{R}{r}\right)^{n+3} P_n (\cos \psi)$$

$$\frac{\partial \Delta g_i}{\partial r} = - \frac{\mu}{R^3} \sum_n (n+2)(n-1) \left(\frac{r}{R}\right)^{n-2} P_n (\cos \psi)$$

Hence, for $r = r_0$:

$$(\delta g_r)_n = \left(\frac{\partial \Delta g_e}{\partial r}\right)_n - \left(\frac{\partial \Delta g_i}{\partial r}\right)_n = \left[\left(\frac{R}{r_0}\right)^{2n+1} - 1 \right] \left(\frac{\partial \Delta g_i}{\partial r}\right)_n$$

where

$$\left(\frac{\partial \Delta g_i}{\partial r}\right)_n = - \frac{\mu}{R^3} (n-1)(n+2) \left(\frac{r_0}{R}\right)^{n-2} P_n (\cos \psi)$$

We notice that the relative error of $\dfrac{\partial \Delta g}{\partial r}$ is of the same order of magnitude as that for T. Thus the spherical harmonic series for $\dfrac{\partial \Delta g}{\partial r}$ is convergent for $R = r_0$ (cf. the gravity anomalies). For this example, we have the relation:

$$(\delta g_r)_n = - \frac{(n-1)(n+2)}{r^2} \delta T_n$$

4. The Error of the Potential

We are now going to estimate the errors of extending the potential series (2.1) to the surface of the earth. Some error estimates of this type were given by Cook (1967) and Levallois (1969). See formulae (4.5 a-b) and below. See also section 4.3.

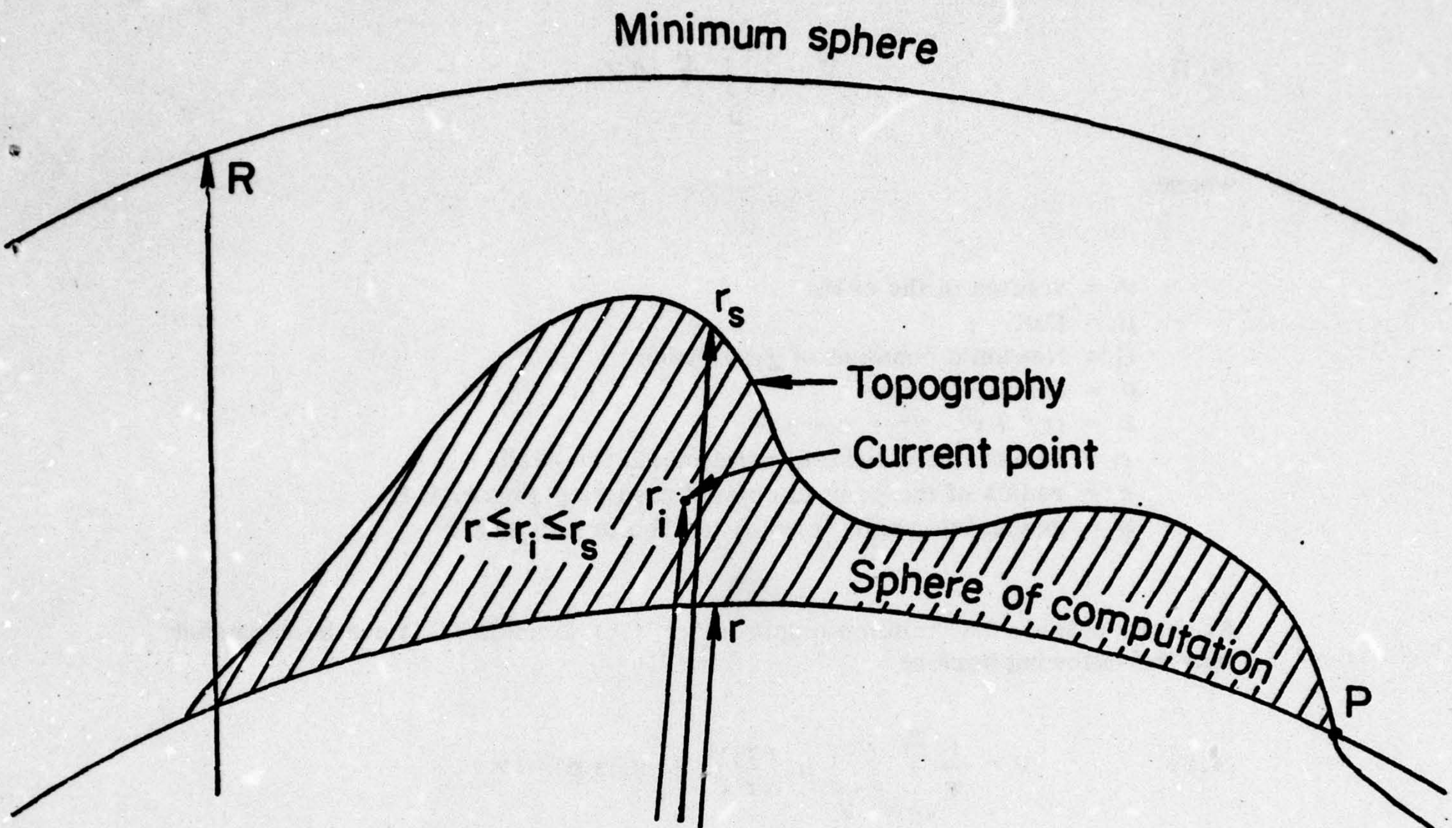

Figure 4.1. The potential is computed at a point P of distance r from the earth's center. The sphere of radius r is denoted the sphere of computation. The topographical masses between this sphere and the minimum sphere makes the spherical harmonic expansion V_e divergent at P.

The Newtonian potential of the earth at an arbitrary point is:

$$V = \iiint_{\nu} \frac{\mu}{\ell} \, d\nu \tag{4.1}$$

where

ν = volume of the earth
μ = $G\rho$
G = Newton's constant of gravitation
ρ = density of mass
ℓ = $(r_1^2 + r^2 - 2r_1 r \cos \psi)^{\frac{1}{2}}$
r_1 = radius of the current point inside the earth
r = radius of the point of computation (See Figure 4.1)
ψ = geocentric angle between the vectors $\bar{r}_1$ and $\bar{r}$

At points outside the "minimum sphere" ($r > R$) formula (4.1) can be expanded into the following series:

$$V = \frac{1}{r} \sum_{n=0}^{\infty} \iiint_{\nu} \mu \left(\frac{r_1}{r}\right)^n P_n (\cos \psi) \, d\nu \tag{4.2}$$

The corresponding convergent series for points inside the minimum sphere ($r < R$) is:

$$V_i = \frac{1}{r} \sum_{n=0}^{\infty} \iint_{\sigma} \left[\int_0^r \mu \left(\frac{r_1}{r}\right)^n + \int_r^{r_s} \mu \left(\frac{r}{r_1}\right)^{n+1} \right] P_n (\cos \psi) \, d\nu \tag{4.3}$$

where r_s is the radius of a current point at the surface of the earth ($r \leq r_1 \leq r_s$) and σ is the unit sphere. See Figure 4.1.

The error δV of extending formula (4.2) to a point inside the minimum sphere is given by the difference between (4.2) and (4.3):

$$\delta V = \sum_{n=0}^{\infty} \delta V_n \tag{4.4}$$

where

$$(4.5a)\qquad \delta V_n = \frac{1}{r}\iint\limits_{\sigma}\int_r^{r_s} \mu\left[\left(\frac{r_1}{r}\right)^n - \left(\frac{r}{r_1}\right)^{n+1}\right] P_n(\cos\psi)\, d\nu\ ,\ r_s \le R$$

or

$$(4.5b)\qquad \delta V_n = \frac{1}{r}\iint\limits_{\sigma}\int_r^{r_s} \mu\left(\frac{r}{r_1}\right)^{n+1}\left[\left(\frac{r_1}{r}\right)^{2n+1} - 1\right] P_n(\cos\psi)\, d\nu$$

Formulae (4.5a-b) were derived by Cook (1967) and Levallois (1969). Cook drew the conclusion that this error is in the order of J_n^3 where J_n is the n-th zonal harmonic of the earth's gravity field. This result can easily be combined with Kaula's rule of thumb (1.1) for estimating the error of a series of spherical harmonics. Even if the series were extended to infinity, the error would be negligible, so that " the 'satellite geoid' is a close enough approximation to the true geoid". However, it is not at all obvious that δV_n according to (4.5 a-b) is of order J_n^3. Levallois (1969) came to a different result, which is reported in section 4.3.

In this section and section 4.1 we are going to develop a formula for δV_n which can be used in a numerical integration. From Heiskanen and Moritz, 1967, p. 33 we obtain:

$$(4.6)\qquad P_n(\cos\psi) = \frac{1}{2n+1}\sum_{m=0}^{n}\left[\bar{R}_{nm}(\theta,\lambda)\,\bar{R}_{nm}(\theta_1,\lambda_1) + \bar{S}_{nm}(\theta,\lambda)\,\bar{S}_{nm}(\theta_1,\lambda_1)\right]$$

where

$\bar{R}_{nm}$, $\bar{S}_{nm}$ = fully normalized spherical harmonics.

We have

$$(4.6a)\qquad \iint \bar{R}_{nm}\ \bar{S}_{n'm'}\ d\sigma = 0 \qquad \text{for all } n,\ n',\ m \text{ and } m'$$

and

$$(4.6b)\qquad 1/4\pi \iint U_{nm}\ U_{n'm'}\ d\sigma = \delta_{nn'}\ \delta_{mm'}$$

where U_{nm} is any of the harmonics $\bar{R}_{nm}$ or $\bar{S}_{nm}$ and δ is the Kronecker's delta.

From formulae (4.5a) and (4.6a-b) we obtain:

(4.7a) $$\delta V_n = \sum_{m=0}^{n} [\, \bar{a}_{nm} \bar{R}_{nm}(\theta,\lambda) + \bar{b}_{nm} \bar{S}_{nm}(\theta,\lambda)]$$

where

(4.7b)

$$\begin{Bmatrix} \bar{a}_{nm} \\ \bar{b}_{nm} \end{Bmatrix} = \frac{1}{(2n+1)\, r^{n+1}} \iint_{\sigma} \int_{r}^{r_s} \mu\, r_1^{n} \left[1 - \left(\frac{r}{r_1} \right)^{2n+1} \right] \begin{Bmatrix} \bar{R}_{nm}(\theta_1,\lambda_1) \\ \bar{S}_{nm}(\theta_1,\lambda_1) \end{Bmatrix} d\nu$$

If we assume that $\mu = \mu(\theta,\lambda)$ (independent of r), we obtain:

(4.8a) $$\begin{Bmatrix} \bar{a}_{nm} \\ \bar{b}_{nm} \end{Bmatrix} = \frac{1}{2n+1} \iint_{\sigma} \mu\, I(r, r_s) \begin{Bmatrix} \bar{R}_{nm} \\ \bar{S}_{nm} \end{Bmatrix} d\sigma$$

where

(4.8b) $$I(r, r_s) = r^2 \begin{cases} 0 & \text{if } r \geq r_s \\ \dfrac{(r_s/r)^{n+3} - 1}{n+3} + \dfrac{(r_s/r)^{-(n-2)} - 1}{n-2} & \text{if } r < r_s,\ n \neq 2 \\ \dfrac{(r_s/r)^5 - 1}{5} - \ln(r_s/r) & \text{if } r < r_s,\ n = 2 \end{cases}$$

Formulae (4.8a-b) can be applied with $\mu = \mu_0$ = constant if we neglect the ellipticity of the earth (spherical earth with topography). The integration is then performed over the topographical masses. If we also consider the earth's ellipticity the integration will include the masses of the oceans above the lower bound r of the integration. See Figure 4.2. If we assume that the oceans have the constant density μ_w and the solid crust the constant density μ_0 the integral (4.7b) becomes (see Figure 4.3):

(4.8c) $$\begin{Bmatrix} \bar{a}_{nm} \\ b_{nm} \end{Bmatrix} = \frac{1}{2n+1} \iint_{\sigma} \overline{\mu\, I(r, r_s)} \begin{Bmatrix} \bar{R}_{nm}(\theta,\lambda) \\ \bar{S}_{nm}(\theta,\lambda) \end{Bmatrix} d\sigma$$

where

$$\overline{\mu I(r, r_s)} = \mu_0 \bar{I}(r, r_b) + \mu_w \bar{I}(r_b, r_s)$$

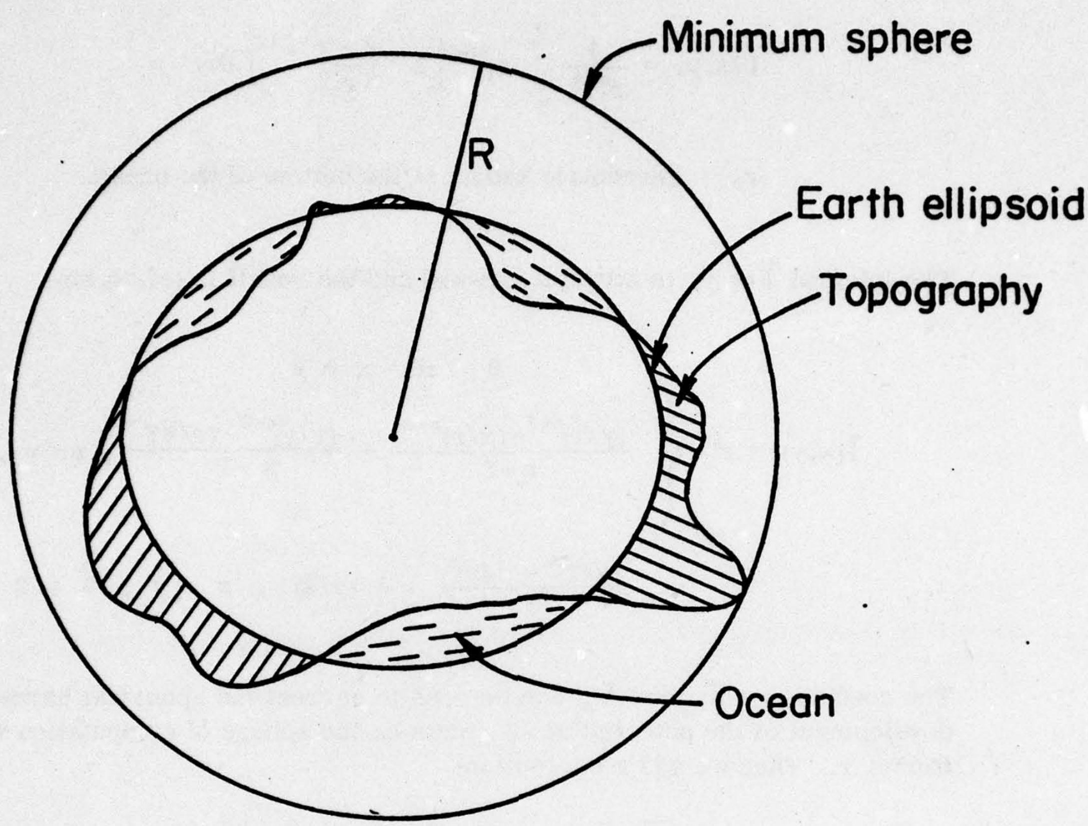

Figure 4.2. The computation of the error coefficients a_{nm} and b_{nm} for any point P on a selected sphere of radius r (sphere of computation) includes the integration of all masses of the oceans and topography between the spheres of radii r and R.

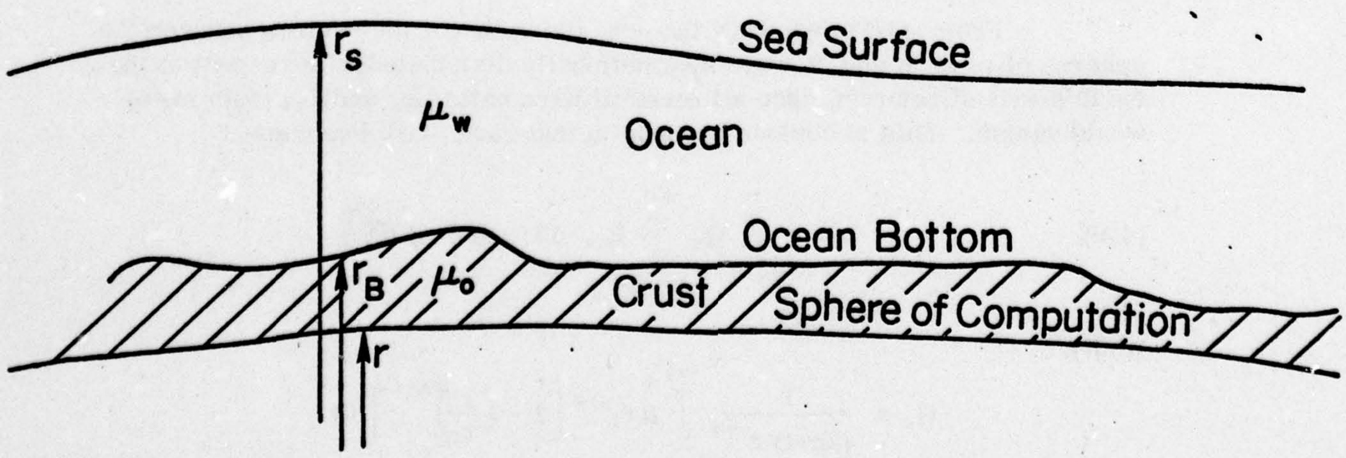

Figure 4.3. The density of the masses above the sphere of computation is μ_o for the solid crust and μ_w for the ocean.

$$\bar{I}(x,y) = \frac{1}{r^{n+1}} \int_x^y r_1^{n+2} \left[1 - \left(\frac{r}{r_1}\right)^{2n+1}\right] dr_1$$

r_b = geocentric radius of the bottom of the ocean.

The integral $\bar{I}(x,y)$ is straight forward and the result is (cf. 4.8b):

$$\bar{I}(x,y) = r^2 \begin{cases} 0 & \text{if} \quad x \geq y \\ \dfrac{(y/r)^{n+3} - (x/r)^{n+3}}{n+3} + \dfrac{(r/y)^{n-2} - (r/x)^{n-2}}{n-2}, & x < y\ ,\ n \neq 2 \\ \dfrac{(y/r)^5 - (x/r)^5}{5} - \ln(y/x), & x < y\ ,\ n = 2 \end{cases}$$

The coefficients $\bar{a}_{nm}$ and $\bar{b}_{nm}$ can be used to correct the spherical harmonic development of the potential[+] at all points on the sphere of computation with radius r. Then we add a correction:

$$\begin{Bmatrix} \Delta\bar{C}_{nm} \\ \Delta\bar{S}_{nm} \end{Bmatrix} = - \frac{R}{GM} \begin{Bmatrix} \bar{a}_{nm} \\ \bar{b}_{nm} \end{Bmatrix} \left(\frac{r}{R}\right)^{n+1}$$

to each potential spherical harmonic coefficient of the series expansion. It should be emphasized that these corrections are valid only for the potential and cannot be used for improving near surface expansions of gravity anomalies (cf. section 6).

From (4.5 a) we draw the conclusion that if the masses between the spheres of radii r and R were symmetrically distributed with respect to the earth's axis of rotation, then all tesseral harmonics $\bar{a}_{nm}$ and $\bar{b}_{nm}$ (with $m \neq 0$) would vanish. This is obvious because in that case (4.7b)becomes:

(4.9) $$\bar{a}_{nm} = \int_{\theta=0}^{\pi} Q_n \int_{\lambda=0}^{2\pi} \bar{R}_{nm}\, d\lambda\ \sin\theta\ d\theta$$

where

$$Q_n = \frac{1}{(2n+1)\, r^{n+1}} \int_r^{r_s} \mu\, r_1^{n+2} \left[1 - \left(\frac{r}{r_1}\right)^{2n+1}\right] dr_1$$

[+] See formula (2.1).

and

$$\int_{\lambda=0}^{2\pi} \bar{R}_{nm}\, d\lambda = 0 \qquad \text{for} \quad m \neq 0$$

The same is true for $\bar{b}_{nm}$. However, for the real earth with its irregular mass distributions this assumption is not valid.

4.1 An Approximate Formula

Formula (4.8b) can be approximated in the following way (for $H = r_s - r > 0$):

$$\left(\frac{r_s}{r}\right)^{n+3} = \left(1+\frac{H}{r}\right)^{n+3} = \sum_{k=0}^{\infty} \binom{n+3}{k} \left(\frac{H}{r}\right)^k =$$

$$= 1 + (n+3)\frac{H}{r} + \frac{(n+3)(n+2)}{2}\left(\frac{H}{r}\right)^2 + \frac{(n+3)(n+2)(n+1)}{1\times2\times3}\left(\frac{H}{r}\right)^3 + \frac{(n+3)(n+2)(n+1)n}{2\times3\times4}\left(\frac{H}{r}\right)^4 + \ldots$$

and

$$\left(\frac{r_s}{r}\right)^{-(n-2)} = \sum_{k=0}^{\infty} \binom{-(n-2)}{k} \left(\frac{H}{r}\right)^k =$$

$$= 1 - (n-2)\frac{H}{r} + \frac{(n-2)(n-1)}{2}\left(\frac{H}{r}\right)^2 - \frac{(n-2)(n-1)n}{1\times2\times3}\left(\frac{H}{r}\right)^3 + \frac{(n-2)(n-1)n(n+1)}{2\times3\times4}\left(\frac{H}{r}\right)^4 + \ldots$$

Inserting these series expansions into (4.8b) we obtain:

$$\text{(4.10)} \qquad I(r, r_s) = (2n+1)\, H^2 \left[\frac{1}{2} + \frac{1}{3}\frac{H}{r} + \frac{n(n+1)}{2\times3\times4}\left(\frac{H}{r}\right)^2 + \ldots\right]$$

In Table 4.1 we compare formula (4.8b) with the approximation $(2n+1)\,H^2/2$ according to formula (4.10). It is shown that the error is increasing with the degree (n). For $n = 50$ the error is less than 1% and for $n = 100$ it is almost 2%.

Table 4.1

Comparison between $(2n+1)\, H^2/2$ and $I(r, R)$
$R = 6384$ km, $H = R - r = 27$ km. Units : km^2

n	$(2n+1)\, H^2/2$	$I(r, R)$	Diff.	Diff./I%
10	7654.5	7677.4	22.9	0.3
30	22234.5	22328.6	94.1	0.4
50	36814.5	37060.1	245.6	0.7
100	73264.5	74591	1327	1.8
150	109714.5	113812	4097	3.6
200	146164.5	155627	9462	6.1
250	182614	201019	18404	9.2

Inserting (4.10) into (4.8a) and neglecting terms of order $\frac{H}{r}$ and higher we finally arrive at:

$$\left\{\begin{matrix} \bar{a}_{nm} \\ \bar{b}_{nm} \end{matrix}\right\} = \frac{1}{2} \iint_{\sigma} \mu \, H^2 \left\{\begin{matrix} \bar{R}_{nm} \\ \bar{S}_{nm} \end{matrix}\right\} d\sigma \qquad (4.11)$$

where

$$H = \begin{cases} 0 & \text{if} \quad r \geq r_s \\ r_s - r & \quad\; r < r_s \end{cases}$$

If we also consider the ellipticity of the earth we obtain from (4.8c):

$$\left\{\begin{matrix} \bar{a}_{nm} \\ \bar{b}_{nm} \end{matrix}\right\} = \tfrac{1}{2} \iint \left[(\mu_O - \mu_w)\, H_b^2 + \mu_w \, H^2 \right] \left\{\begin{matrix} \bar{R}_{nm} \\ \bar{S}_{nm} \end{matrix}\right\} d\sigma \qquad (4.11a)$$

where

μ_O = density of the solid crust
μ_w = density of the oceans

$$H_b = \begin{cases} H \text{ if continent} \\ 0 \text{ if ocean with } r \geq r_b \\ r_b - r \text{ otherwise.} \end{cases}$$

4.2 Effect of Ellipticity

In this section we are going to study the error of extending the spherical harmonic expansion of the potential to a point P at the surface of an oblate homogeneous ellipsoid (see Figure 4.4). The error is caused by the masses outside the sphere of computation of radius r.

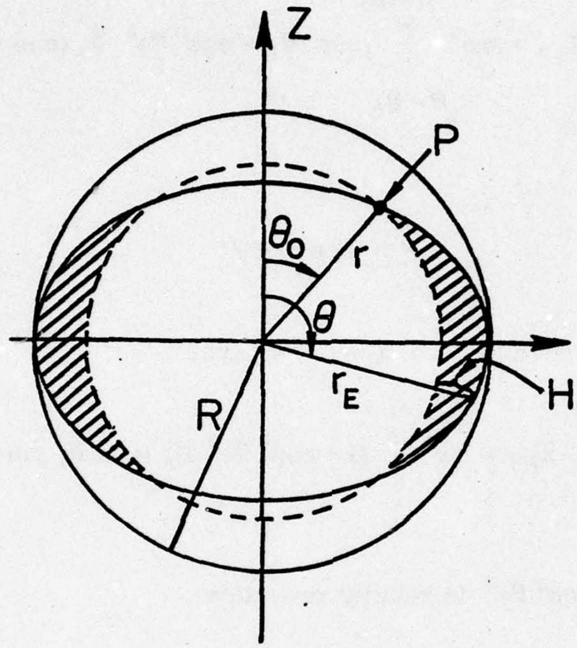

Figure 4.4. The downward continuation error at P is obtained by integration over the (shaded) masses between the spheres of radii r and R.

The error coefficients due to the improper downward continuation can be determined by formula (4.11). With notations according to Figure 4.4, we obtain:

$$r_E = R\sqrt{\frac{1-e^2}{1-e^2\sin^2\theta}} \quad , \quad r = R\sqrt{\frac{1-e^2}{1-e^2\sin^2\theta_0}}$$

and

$$H = r_E - r = \frac{Re^2}{2} (\cos^2 \theta_0 - \cos^2 \theta) + O(e^4)$$

It has already been shown in (4.9) that the coefficients $\bar{a}_{nm}$ and $\bar{b}_{nm}$ are 0 for $m \neq 0$. For $m = 0$ we obtain from (4.11) and the above expression for H, when neglecting terms of order higher than e^4:

(4.12a) $$\bar{a}_{n0} = c e^4 \int_{\theta = \theta_0}^{\pi - \theta_0} (\cos^2 \theta_0 - \cos^2 \theta)^2 \, \bar{P}_n (\cos \theta) \sin \theta \, d\theta$$

where

$$c = G \rho \pi R^2/4$$

In the special case $\theta_0 = 0$ we have:

(4.12b) $$\bar{a}_{n0} = c e^4 \int_0^{\pi} (1 - \cos^2 \theta)^2 \, \bar{P}_n (\cos \theta) \sin \theta \, d\theta$$

The factor $(1 - \cos^2 \theta)^2$ is readily rewritten:

$$(1 - \cos^2 \theta)^2 = c_0 + c_2 \bar{P}_2 (\cos \theta) + c_4 \bar{P}_4 (\cos \theta)$$

where

$$c_0 = \frac{8}{15}, \qquad c_2 = -\frac{16}{21\sqrt{5}}, \qquad c_4 = \frac{8}{105}$$

Inserting this expression into (4.12b) and using the orthogonality property of the Legendre's polynomials:

$$\int_{-1}^{1} \bar{P}_n (t) \, \bar{P}_m (t) \, dt = \begin{cases} 2 & n = m \\ 0 & n \neq m \end{cases}$$

we finally arrive at

$$\bar{a}_{n0} = 2ce^4 \begin{cases} c_0 & \text{if} \quad n = 0 \\ c_2 & \text{if} \quad n = 2 \\ c_4 & \text{if} \quad n = 4 \\ 0 & \text{otherwise} \end{cases}$$

For $\rho = 2.67 g/cm^3$, $R = 6378$ km, $r_0 = 6371$ km, $\gamma = 978$ gal and $e^2 = 0.0067$ we obtain:

$$\frac{\bar{a}_{00}}{\gamma} = 27.9 \text{ m} \qquad \left| \frac{\bar{a}_{00}}{\gamma r_0} \right| \approx 4.3 \times 10^{-6}$$

$$\frac{\bar{a}_{20}}{\gamma} = -17.8 \text{ m} \qquad \left| \frac{\bar{a}_{20}}{\gamma r_0 \bar{C}_{20}} \right| \approx 5.8 \times 10^{-3}$$

$$\frac{\bar{a}_{40}}{\gamma} = 4.0 \text{ m} \qquad \left| \frac{\bar{a}_{40}}{\gamma r_0 \bar{C}_{40}} \right| \approx 1.1$$

From these coefficients we obtain the following downward continuation error at the poles ($\theta_0 = 0$):

$$\delta V = \sum_{n=0}^{4} \bar{a}_{n0} \sqrt{2n+1} = 0$$

Thus the errors of the individual spherical harmonics compensate each other at the poles.

For an arbitrary point $P(\theta_0)$ on the surface of the ellipsoid the downward continuation error in an expansion to degree N is given by:

$$\delta V(N) = \sum_{n=0}^{N} \bar{a}_{n0}(r) \bar{P}_n(\cos\theta_0)$$

Inserting (4.12a) into this formula and changing the order of summation and integration we obtain:

$$\delta V(N) = c e^4 \int_{\theta_0}^{\pi - \theta_0} (\cos^2\theta_0 - \cos^2\theta)^2 \sum_{n=0}^{N} \overline{P}_n(\cos\theta)\,\overline{P}_n(\cos\theta_0) \sin\theta \, d\theta$$

Using the substitutions:

$$t = \cos\theta \quad , \qquad t_0 = \cos\theta_0$$

and

$$\overline{P}_n(t)\,\overline{P}_n(t_0) = (2n+1)\, P_n(t)\, P_n(t_0)$$

we arrive at

$$\delta V(N) = c e^4 \int_{-t_0}^{t_0} (t_0^2 - t^2)^2 \sum_{n=0}^{N} (2n+1)\, P_n(t)\, P_n(t_0)\, dt \qquad (4.13)$$

It is shown in Proposition 4 of the Appendix that this formula may be written:

$$\delta V(N) = c\, e^4\, t_0^2\, g_N(\overline{t})$$

where

$$g_N(t) = 2\,\frac{M(M+1)}{2M+1}\left[\frac{P_M(t) - P_{M-2}(t)}{2M-1}\, P_{M+1}(t_0) - \frac{P_{M+2}(t) - P_M(t)}{2M+3}\, P_{M-1}(t_0)\right]$$

$$M = \begin{cases} N \text{ if } N \text{ is odd} \\ N+1 \text{ if } N \text{ is even} \end{cases}$$

and

$$0 \le \overline{t} \le t_0$$

The function g_N (and subsequently $\delta V(N)$) has the following properties:

$$g_N(0) = g_N(1) = 0$$

and

$$\lim_{N\to\infty} g_N = 0$$

From formula (6.9) we obtain:

$$c e^2 g_N(t_o) = \frac{R}{4} \delta \Delta g (N)$$

and the rough approximation:

$$\delta V(N) \approx \frac{Re^2}{4} t_o^2 \delta \Delta g (N) \tag{4.13a}$$

where $\delta \Delta g (N)$ is the downward continuation error of Δg in an expansion to degree N. For $R = 6378$ km, $e^2 = 0.0067$ and $\gamma = 978$ gal we have:

$$\frac{\delta V(N)}{\gamma} \approx 0.011 \; t_o^2 \; \delta \Delta g (N) \text{ meter,}$$

where $\delta \Delta g$ is in units of mgal. From Fig. 6.1 we finally obtain:

$$\frac{\delta V(16)}{\gamma} \leq 0.45 \text{ m} \quad \text{and} \quad \frac{\delta V(45)}{\gamma} \leq 0.19 \text{ m}$$

This development indicates that the downward continuation error of the truncated spherical harmonic expansion of V is small and decreasing with the degree of truncation. On the other hand, the following spherical harmonic expansion for the potential in the exterior of a homogeneous oblate ellipsoid is known from potential theory (see MacMillan, 1958, p. 363):

$$V_e = \frac{3GM}{r} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)} \left(\frac{ae}{r}\right)^{2n} P_{2n} (\cos \theta) \tag{4.14}$$

where M is the mass of the ellipsoid. This development has the radius of convergence ae, which is inside the ellipsoid. However, the potential provided by (4.14) is correct only on the surface and outside the ellipsoid. This fact is indirectly verified by letting e approach zero, in which case (4.14) becomes:

$$V_e = \frac{GM}{r}$$

This is the well known exterior potential of a homogeneous sphere. The radius of convergence is zero.

In conclusion, formula (4.13) is the difference between the expansion (4.14) to degree N and a spherical harmonic expansion (at the radius r), which is valid also inside the surface of matter. For N approaching infinity, the sums of the two series are identical for exterior points. Inside the surface of matter they differ.

In geodesy, we are mainly interested in the exterior gravity field and a potential expansion of the mean earth ellipsoid (MEE) similar to (4.14) is therefore most convenient. Unfortunately, this ellipsoid is not homogeneous and (4.14) can not be applied. However, the MEE is currently approximated by a level ellipsoid, the external gravitational potential of which can be expanded in the following way [Heiskanen and Moritz, 1967, formulae (2-88) and (2-91)]:

$$V_e = \frac{3GM}{r} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+3)} \left(1 - n + 5n \frac{C-A}{Ma^2e^2}\right) \left(\frac{ae}{r}\right)^{2n} P_{2n}(\cos\theta)$$

where C and A are the principal moments of inertia. If we anticipate this model for the mean earth ellipsoid, there will be no contribution to the downward continuation error from the ellipsoid. However, the previously derived correction formulae (4.8 a-b) and (4.11) for the disturbing topography needs a modification. In these formulae it was assumed that the radius (r_m) of the mean sea level was constant. Now we assume that:

$$r_m = R\sqrt{\frac{1-e^2}{1-e^2\sin^2\theta}}$$

Then (4.8a-b) and (4.11) are modified by inserting:

$$r = \begin{cases} r_m & \text{if } r < r_m \\ r & \text{if } r \geq r_m \end{cases} \tag{4.15}$$

Formulae (4.8a-b), (4.11) and (4.15) should be feasible expressions for a numerical study of the downward continuation error of the potential. Formulae (4.8c) and (4.11a) are less convenient unless the spherical harmonic expansion that is valid at the entire sphere of computation (also inside the earth) is required.

4.3 An Error Estimate for the Potential

We may substitute δV of formulae (4.4) through (4.7) by the error of the disturbing potential, δT, if we assume that the reference field is unchanged. Formula (4.5b) is then written:

$$\delta T_n = \frac{1}{r} \iint_{\sigma} \int_{r}^{r_s} \mu \left(\frac{r}{r_1}\right)^{n+1} \left[\left(\frac{r_1}{r}\right)^{2n+1} - 1\right] P_n(\cos\psi)\, d\nu \tag{4.16}$$

At sea level ($r = r_0 = R-H$) this error is of the following order of magnitude:

$$\delta T_n \approx \left[\left(\frac{R}{r_0}\right)^{2n+1} - 1\right] \Delta T_n \tag{4.17a}$$

where

$$\Delta T_n = \frac{1}{r_0} \iint \int_{r_0}^{r_s} \mu \left(\frac{r_0}{r_1}\right)^{n+1} P_n(\cos\psi)\, d\nu \tag{4.17b}$$

For low degrees of n we may use the following approximation:

$$\delta T_n \approx (2n+1) \frac{H}{R} \Delta T_n \tag{4.18}$$

ΔT_n is the contribution to T_n from the topography. If we assume that the disturbing potential is generated entirely within the topography of the earth, we have [cf. formula (4.3)]:

$$\Delta T_n \approx T_n \tag{4.19}$$

Assuming that the low order harmonics of T is only partly originated within the topography, we may use the following approximation:

$$\Delta T_n = p_n T_n \tag{4.20}$$

where

$$p_n = 1 - e^{-\alpha n} \quad , \quad 0 < \alpha < \infty$$

The coefficient α can be determined empirically by comparing a spherical harmonic expansion of ΔT (formula 4.17b) with that of T. In the limit $\alpha = \infty$, (4.20) is identical with (4.19). For $\alpha = 0$, we obtain $\Delta T_n = 0$, that is, no contribution to T_n from the topography. The approximations (4.18) and (4.19) were used by Levallois (1969) in the following way:

$$\delta T = \sum_n \delta T_n \approx \frac{H}{R} \sum_n (2n+1) \; T_n =$$

$$= H \sum_n \frac{2n+1}{n-1} \; \Delta g_n \approx 2H \; \Delta g$$

where we have used the relation:

$$T_n = \frac{R \Delta g_n}{n-1}$$

The error of the geoidal height is then:

$$\delta N = \frac{\delta T}{\gamma} \approx 2H \frac{\Delta g}{\gamma}$$

For $H = 27$ km, $\Delta g = 100$ mgal and $\gamma = 980$ gal this formula implies:

$$\delta N \approx 5\text{m}$$

However, the approximation of (4.17a) by (4.18) is adequate only for lower orders, (say, $n < 150$). Consequently, the above order of the error does not hold if we include harmonics to infinity.

4.4 Global RMS Errors

Spherical harmonics of different degrees and/or orders are orthogonal to each other, see formulae (4.6a-b).

Thus, the global RMS error of δT is given by:

$$\|\delta T\| = \left[\frac{1}{4\pi}\sum_{n}\iint_{\sigma}(\delta T_n)^2\, d\sigma\right]^{\frac{1}{2}} \tag{4.21a}$$

where

$$\frac{1}{4\pi}\iint(\delta T_n)^2\, d\sigma = \|\delta T_n\|^2 = \sum_{m=0}^{n}(\bar{a}_{nm}^2 + \bar{b}_{nm}^2) \tag{4.21b}$$

The last formula is obtained by substituting $(\delta V)_n$ by $(\delta T)_n$ in formula (4.7a). The total error of T for a truncated series of spherical harmonics is (cf. section 3):

$$\epsilon_T(N) = \delta T(N) + e_T(N) \tag{4.22}$$

where N is the order of truncation, $\delta T(N)$ is the downward continuation error and $e_T(N)$ is the truncation error. We obtain:

$$\|e_T(N)\| = \left[\frac{1}{4\pi}\sum_{N+1}^{\infty}\iint T_n^2\, d\sigma\right]^{\frac{1}{2}} = \left[\sum_{N+1}^{\infty}\sigma_n^2\right]^{\frac{1}{2}} \tag{4.23}$$

where

$$\sigma_n^2 = \frac{1}{4\pi}\iint_{\sigma} T_n^2\, d\sigma$$

σ_n^2 are the degree variances of the potential. From (4.22) we arrive at the following formula for the total RMS error when noting the orthogonality between spherical harmonics of different degrees:

$$\|\epsilon_T(N)\|^2 = \|\delta T(N)\|^2 + \|e_T(N)\|^2$$

Definition 4.1: The optimum degree of truncation (N_{opt}) implies

$$\|\epsilon_T(N)\| = \text{minimum}$$

The downward continuation error is increasing with N while the truncation error is decreasing. Consequently, $\|\epsilon_T(N)\|$ is minimum when the Nth term of $\|\delta T(N)\|$ equals the (N+1)th term of $\|e_T(N)\|$, i.e.:

$$\|\delta T_N\| = \sigma_{N+1} \longleftrightarrow N = N_{Opt} \tag{4.24}$$

An approximation of δT_N is obtained from (4.17a) and (4.19):

$$\delta T_n \approx \left[\left(\frac{R}{r_O}\right)^{2n+1} - 1\right] T_n \tag{4.25}$$

where R is the radius of the minimum sphere ($\sim$ 6384 km) and r_O is the mean earth radius ($\sim$ 6370 km). Inserting (4.25) into (4.24) we obtain:

$$\left(\frac{R}{r_O}\right)^{2N+1} - 1 = \frac{\sigma_{N+1}}{\sigma_N}$$

As the degree variances are decreasing with N we easily arrive at:

$$N_{Opt} \leq \frac{r_O \ln 2}{2H}, \quad H = r - r_O \tag{4.26}$$

or with the above numerical values for R and r_O:

$$N_{Opt} \leq 158$$

The following values for the potential degree variances were given by Tscherning and Rapp (1974):

$$\sigma_n^2 = \begin{cases} 0 & \text{for } n \leq 2 \\ \dfrac{A\, r_O^2}{(n-2)(n-1)(n+24)}\, s^{n+1} & n > 2 \end{cases}$$

where

$s = (r_{Bje}/r_O)^2 = 0.999617$
$A = 425.18$ [mgal^2]
$r_O = 6371$ [km] ($r_O = 6370$ km was used in the computations.)

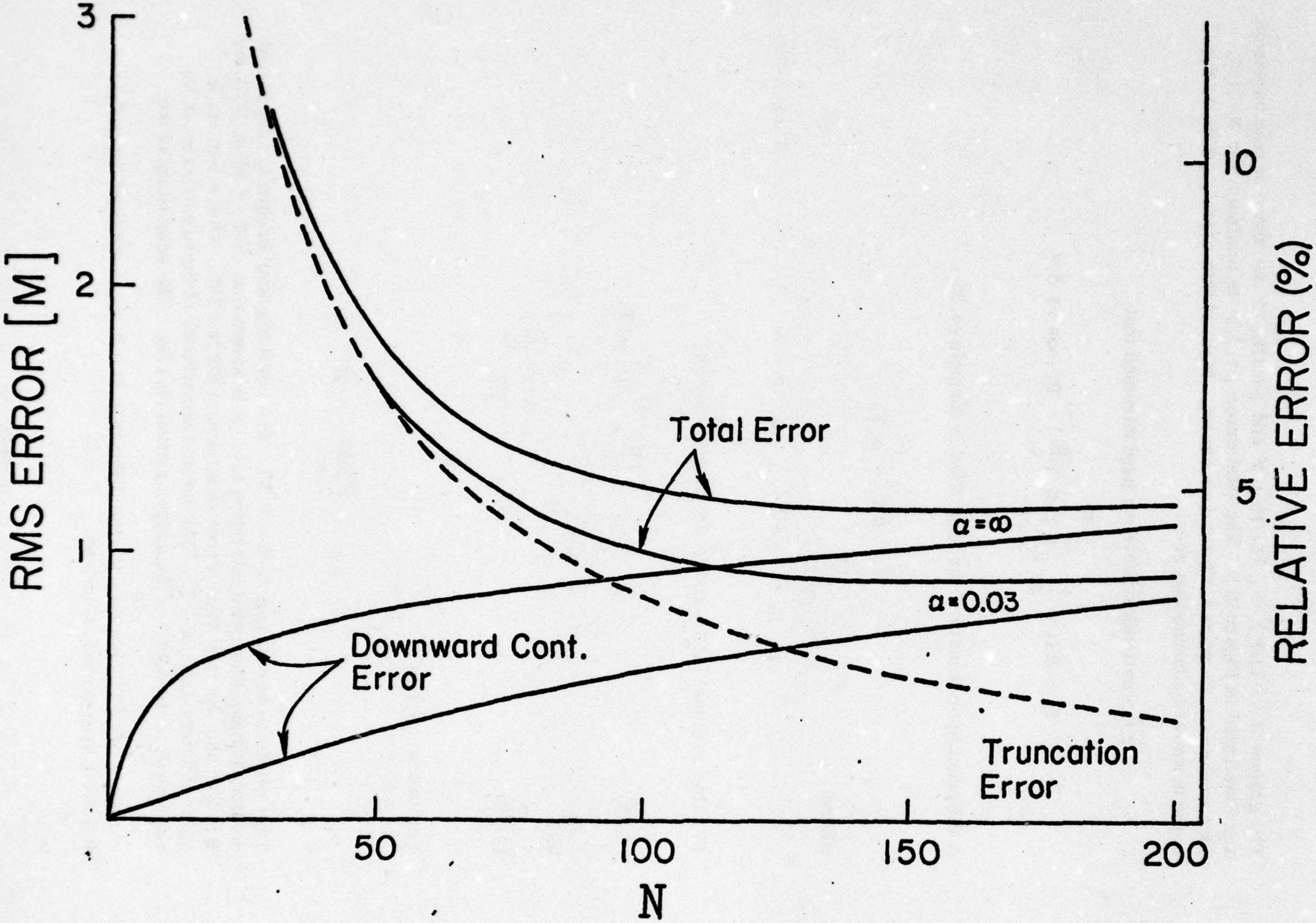

Figure 4.5. The RMS errors of the downward continuation and truncation of the geoidal undulation expansion to degree N. R = 6384 km and r_0 = 6370 km.

The graphs of $\|\delta T(N)\|/\gamma$, $\|e_T(N)\|/\gamma$ and $\|\epsilon_T(N)\|/\gamma$ for these degree variances are illustrated in Figure 4.5. The total error $\|\epsilon_T\|/\gamma$ is minimum for N = 157. This error is 1.17 m ($\approx$ 4.7% relative error). Most of this error (1.05m) is due to the downward continuation error.

In the estimates above we have assumed that:

$$T_n \approx \Delta T_n = \frac{1}{r_0} \iint_{\sigma} \int_{r_0}^{r_s} \mu \left(\frac{r_0}{r_1}\right)^{n+1} P_n(\cos\psi)\, d\nu$$

A different approximation was suggested in formula (4.20):

$$\Delta T_n = p_n T_n$$

where

$$p_n = 1 - e^{-\alpha n} \quad , \quad 0 < \alpha < \infty$$

For this estimate (4.25) and (4.26) take the forms:

$$\delta T_n = \left[\left(\frac{R}{r_0}\right)^{2n+1} - 1 \right] p_n T_n \tag{4.25'}$$

and

$$N_{opt} \leq \frac{r_0 \ln(1 + \frac{1}{p_n})}{2H} \tag{4.26'}$$

Let us assume that:

$$p_{100} = \frac{\Delta T_{100}}{T_{100}} = 0.95$$

This relation corresponds to $\alpha = 0.03$. The resulting RMS errors for downward continuation are illustrated in Figure 4.5. It is shown that $\|\delta T\| < 4\%$ of $\|T\|$ for $N < 200$ and the total RMS error is minimum for N = 158. This minimum is approximately 3.5% of $\|T\|$. The largest uncertainty of these estimates is the relation $\frac{R}{r_0}$ introduced in the approximation (4.17a). The sensitivity of the solutions of $\|\delta T\|$ and N_{Opt} for the difference $H = R - r_0$ is obtained from the following approximate solutions:

$$\delta T(N) \approx \frac{H}{R} \sum_{n=0}^{N} (2n+1)\, T_n$$

and

$$N_{Opt} \approx \frac{r_0 \ln 2}{2H}$$

Differentiating this formulae with respect to H we arrive at:

$$\left|\frac{d\,\delta T(N)}{\delta T(N)}\right| = \left|\frac{dN_{Opt}}{N_{Opt}}\right| = \left|\frac{dH}{H}\right| \tag{4.27}$$

Now H is in the order of 13 km and the error dH may be several kilometers. If we assume that $dH \approx 4$ km, we obtain a relative error of 31%. Thus we conclude that the RMS estimates given in this section might have considerable errors due to the uncertainty of the basic approximation (4.17a).

5. The Error of Gravity Disturbances

The error of downward continuation of the vertical derivative of V in a series expansion is obtained from formulae (4.2) through (4.4):

$$\delta \frac{\partial V}{\partial r} = \frac{\partial V_e}{\partial r} - \frac{\partial V_i}{\partial r} = \frac{\partial}{\partial r}\, \delta V$$

If we subtract the normal field U from V in this differentiation (and change sign of the whole expression) we obtain the gravity disturbance:

$$\delta_r = -\frac{\partial}{\partial r}(V - U) = -\frac{\partial}{\partial r} T$$

Assuming that the normal field is the same for $T_e (= V_e - U)$ and $T_i (= V_i - U)$ we have the error:

$$\Delta \delta_r = -\frac{\partial}{\partial r}\, \delta T = -\frac{\partial}{\partial r}\, \delta V \tag{5.1}$$

where according to formula (4. 4) and (4. 5a):

$$\delta V = \frac{1}{r} \sum_{n=0}^{\infty} \iint_{\sigma} \int_{r}^{r_s} \mu \left[\left(\frac{r_1}{r}\right)^n - \left(\frac{r}{r_1}\right)^{n+1} \right] P_n (\cos \psi)\, d\nu$$

We use the notation:

$$\Delta \delta_r = \sum_{n=0}^{\infty} \Delta_n$$

where

$$\Delta_n = - \frac{\partial}{\partial r} \delta V_n$$

It is shown in the Appendix, Proposition 1, that Δ_n can be written:

$$\Delta_n = \frac{1}{r^2} \iint_{\sigma} \int_{r}^{r_s} \mu \left[\left(\frac{r_1}{r}\right)^n (n+1) + \left(\frac{r}{r_1}\right)^{n+1} n \right] P_n (\cos \psi)\, d\nu$$

This expression can be rearranged in the following way:

$$\Delta_n = \frac{n+1}{r^{n+2}} \iint_{\sigma} \int_{r}^{r_s} \mu\, r_1^n \left[1 - \left(\frac{r}{r_1}\right)^{2n+1} \right] P_n (\cos \psi)\, d\nu +$$

$$+ \frac{2n+1}{r^2} \iint_{\sigma} \int_{r}^{r_s} \mu \left(\frac{r}{r_1}\right)^{n+1} P_n (\cos \psi)\, d\nu$$

Comparing with formula (4. 5a) we find:

(5.2) $$\Delta_n = \frac{(n+1)}{r} \delta T_n + \frac{2n+1}{r^2} \iint_{\sigma} \int_{r}^{r_s} \mu \left(\frac{r}{r_1}\right)^{n+1} P_n (\cos \psi)\, d\nu$$

where δT_n is the error of T_n caused by the analytic continuation down to r. Furthermore, we have for $r = r_0$ [radius at mean sea level, cf. (4.17b)]:

$$(5.3) \qquad \Delta T_n = \frac{1}{r_0} \iint_{\sigma} \int_{r_0}^{r_s} \mu \left(\frac{r_0}{r_1}\right)^n P_n (\cos \psi)\, d\nu$$

ΔT_n is the contribution to the disturbing potential harmonic T_n from the anomalous masses located outside the sphere of radius r_0. Thus we have:

$$\Delta_n = \frac{1}{r_0} [(n+1)\ \delta T_n + (2n+1)\ \Delta T_n]$$

or, after inserting (4.17a) for δT_n:

$$(5.4) \qquad \Delta_n = \frac{\Delta T_n}{r_0} \left[(n+1)\left(\frac{R}{r_0}\right)^{2n+1} + n\right]$$

Finally, we sum the harmonics Δ_n to the degree of truncation (N):

$$(5.5) \qquad \Delta \delta_r = \frac{1}{r_0} \sum_{n=0}^{N} \left[(n+1)\left(\frac{R}{r_0}\right)^{2n+1} + n\right] \Delta T_n$$

where according to (4.20):

$$\Delta T_n = p_n T_n \ , \quad p_n = 1 - e^{-\alpha n}$$

Formula (5.2) gives an "exact" value of the downward continuation error for $-\frac{\partial T}{\partial r}$. Formula (5.5) is an approximation.

6. The Error of Gravity Anomalies

The gravity anomalies are related to the gravity disturbances by:

$$\Delta g = \delta_r - \frac{2T}{r}$$

In the same way we obtain for the errors of downward continuation:

$$\delta \Delta g = \Delta \delta_r - \frac{2\,\delta T}{r}$$

or

$$\delta \Delta g_n = \Delta_n - \frac{2\,\delta T_n}{r}$$

where $\delta \Delta g$ is the error of Δg and $\delta \Delta g_n$ is its n-th harmonic. By inserting (5.2) we obtain:

(6.1)
$$\delta \Delta g_n = \frac{n-1}{r}\,\delta T_n + \frac{2n+1}{r^2} \iint_\sigma \int_r^{r_s} \mu \left(\frac{r}{r_1}\right)^{n+1} P_n(\cos\psi)\, d\nu$$

and approximately from (5.4) for $r = r_0$:

(6.1')
$$\delta \Delta g_n = \frac{\Delta T_n}{r_0}\left[(n-1)\left(\frac{R}{r_0}\right)^{2n+1} + n + 2\right]$$

This error is of the same order as Δ_n .

The first part of (6.1) is already expressed in terms of spherical harmonics (see formulae 4.7 a-b). The second term may be written:

(6.2a)
$$\frac{2n+1}{r^2} \iint_\sigma \int_r^{r_s} \mu \left(\frac{r}{r_1}\right)^{n+1} P_n(\cos\psi)\, d\nu = \frac{1}{r} \sum_{m=0}^{n} [\bar{c}_{nm}\,\bar{R}_{nm} + \bar{d}_{nm}\,\bar{S}_{nm}]$$

where

(6.2b)
$$\begin{Bmatrix} \bar{c}_{nm} \\ \bar{d}_{nm} \end{Bmatrix} = \frac{1}{r} \iint_\sigma \int_r^{r_s} \mu \left(\frac{r}{r_1}\right)^{n+1} \begin{Bmatrix} \bar{R}_{nm} \\ \bar{S}_{nm} \end{Bmatrix} d\nu$$

Assuming that $\mu = \mu(\theta, \lambda)$ (independent of r) we obtain:

(6.3a)
$$\left\{\begin{matrix}\bar{c}_{ns}\\ \bar{d}_{ns}\end{matrix}\right\} = \iint \mu \; J(r, r_s) \left\{\begin{matrix}\bar{R}_{ns}\\ \bar{S}_{ns}\end{matrix}\right\} d\sigma$$

where

(6.3b)
$$J(r, r_s) = r^2 \begin{cases} 0 & , \text{ if } r \geq r_s \\ \dfrac{1-(r/r_s)^{n-2}}{n-2} & , \text{ if } r < r_s , \; n \neq 2 \\ \ln(r_s/r) & , \text{ if } r < r_s , \; n = 2 \end{cases}$$

For $H = r_s - r > 0$ the last formula can be expanded into the following series:

(6.3c)
$$J(r, r_s) = r H - \frac{n-1}{2} H^2 + \ldots$$

The error of Δg is thus given by:

(6.4)
$$\delta \Delta g = \sum_{n=0}^{\infty} \sum_{s=0}^{n} [\bar{A}_{ns} \bar{R}_{ns} + \bar{B}_{ns} \bar{S}_{ns}]$$

where

$$\bar{A}_{ns} = \frac{1}{r} \; [(n-1) \; \bar{a}_{ns} + \bar{c}_{ns}]$$

$$\bar{B}_{ns} = \frac{1}{r} \; [(n-1) \; \bar{b}_{ns} + \bar{d}_{ns}]$$

or

(6.5)
$$\left\{\begin{matrix}\bar{A}_{ns}\\ \bar{B}_{ns}\end{matrix}\right\} = \frac{1}{2n+1} \iint \mu \; K(r, r_s) \left\{\begin{matrix}\bar{R}_{ns}\\ \bar{S}_{ns}\end{matrix}\right\} d\sigma$$

where

$$K(r, r_s) = r \begin{cases} 0 \quad \text{if} \quad r \geq r_s \\ (n-1) \dfrac{(r_s/r)^{n+3}-1}{n+3} - (n+2) \dfrac{(r_s/r)^{-(n-2)}-1}{n-2} & \text{if } r < r_s, \; n \neq 2 \\ \dfrac{(r_s/r)^5-1}{5} + 4 \ln(r_s/r) & \text{if } r < r_s , \; n = 2 \end{cases}$$

The last formula may be approximated by:

$$\left\{\begin{matrix}\bar{A}_{nm}\\ \bar{B}_{nm}\end{matrix}\right\} = G\rho_0 \iint H \left\{\begin{matrix}\bar{R}_{nm}\\ S_{nm}\end{matrix}\right\} d\sigma \tag{6.6}$$

For one more term in the approximation (6.6) see Prop. 2 of the Appendix. Formula (6.6) can also be determined directly from the potential coefficients $\bar{a}_{nm}$ and $\bar{b}_{nm}$, using the boundary condition (2.2):

$$\left\{\begin{matrix}\bar{A}_{nm}\\ \bar{B}_{nm}\end{matrix}\right\} = -\left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\left\{\begin{matrix}\bar{a}_{nm}\\ \bar{b}_{nm}\end{matrix}\right\}$$

or

$$K(r, r_s) = -\left(\frac{\partial}{\partial r} + \frac{2}{r}\right) I(r, r_s)$$

Inserting (4.10) and noting that $\frac{\partial}{\partial r} = -\frac{\partial}{\partial H}$ we finally obtain:

$$K(r, r_s) = (2n+1)\, H \left[1 + \frac{(n-1)(n+2)}{2 \times 3}\left(\frac{H}{r}\right)^2 + \dots\right] \tag{6.7}$$

where the first term of K is the same as in (6.6). In Table 6.1 it is shown that (6.6) is a good approximation of (6.5).

Table 6.1

Comparison between $H = 27$ km and $\frac{1}{2n+1} K(r, R)$

for $R = 6384$ km, $r = R - H$

n	$\frac{1}{2n+1} K(r, R)$ [km]	Difference [m]	Rel. diff. %
10	27.00877	- 8.77	0.03
20	27.03394	- 33.94	0.13
50	27.20732	-207.3	0.77
100	27.82721	-827.2	3.06

In formulae (6.3 a-b) and (6.6) we have assumed a spherical earth with topography, where the only contribution to the coefficients $\bar{A}_{nm}$ and $\bar{B}_{nm}$ are given by the topography. We may also include the effect of the earth's ellipticity. In that case $\mu K(r, r_s)$ of (6.5) is replaced by (see Figure 4.3):

$$\mu_o\, K(r, r_b) + \mu_w\, \bar{K}(r_b, r_s)$$

where r_b is the radius of the ocean bottom and:

$$\bar{K}(r_b, r_s) = r \begin{cases} 0 & \text{if} \quad r_b \geq r_s \\ (n-1)\dfrac{(r_s/r)^{n+3} - (r_b/r)^{n+3}}{n+3} - (n+2)\dfrac{(r_s/r)^{-(n-2)} - (r_b/r)^{-(n-2)}}{n-2}, & r_b < r_s,\ n \neq 2 \\ \dfrac{(r_s/r)^5 - (r_b/r)^5}{5} + 4\,\ln\,(r_s/r_b), & r_b < r_s,\ n = 2 \end{cases}$$

In this case we arrive at the following approximate formula corresponding to (6.6):

$$(6.6') \qquad \left\{ \begin{matrix} \bar{A}_{nm} \\ \bar{B}_{nm} \end{matrix} \right\} = \iint_\sigma F(H, H_b) \left\{ \begin{matrix} \bar{R}_{nm} \\ \bar{S}_{nm} \end{matrix} \right\} d\sigma$$

where

$$F(H, H_b) = \mu_w H + (\mu_o - \mu_w) H_b$$

$$H_b = \begin{cases} H & \text{if continent} \\ 0 & \text{if ocean with } r > r_b \\ r_b - r & \text{otherwise} \end{cases}$$

However, in the next section it is shown that formula (6.6´) is not very useful for our purpose to determine the downward extension error to the surface of the earth. It seems more convenient to use a modification of (6.6) [cf. section 4.2].

6.1 Effect of Ellipticity

The error of Δg due to the improper downward continuation to the surface of a homogeneous ellipsoid can be estimated in the same way as the error of V in section 4.2. In this case, we obtain for an arbitrary point $P(\theta_o)$ on the ellipsoid.

(6.8) $$\delta \Delta g\,(N) = \sum_{n=0}^{N} \bar{A}_{no}\, \bar{P}_n\,(\cos \theta_o)$$

where

$$\bar{A}_{no} = k e^2 \int_{\theta_o}^{\pi-\theta_o} (\cos^2\theta_o - \cos^2\theta)\, \bar{P}_n\,(\cos\theta)\, \sin\theta\; d\theta$$

and

$$k = G\rho_o\, \pi\, R$$

For $\theta_o = 0$ the last formula is easily integrated (cf. section 4.2) and the result is:

$$\bar{A}_{no} = 4ke^2 \begin{cases} \frac{1}{3} & \text{for } n = 0 \\ -\frac{1}{3\sqrt{5}} & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

For $\rho_o = 2.67\ g/cm^3$, R = 6378 km and $e^2 = 0.0067$ we have the following numerical values of the error coefficients:

$$\bar{A}_{oo} = 3.188 \text{ gal.} \qquad \text{and} \qquad \bar{A}_{2o} = -1.425 \text{ gal.}$$

Inserting these coefficients into (6.8) we arrive at:

$$\delta \Delta g\,(N) = \sum_{n=0}^{N} \bar{A}_{no}\, \bar{P}_n\,(1) = 0, \quad N \geq 2$$

Thus there is no error at the poles. For an arbitrary point $P(\theta_0)$ we obtain from Proposition 3 of the Appendix:

$$\text{(6.9)} \qquad \delta\Delta g\,(N) = 2ke^2\,\frac{M(M+1)}{2M+1}\left[\frac{P_M - P_{M-2}}{2M-1}\,P_{M+1} - \frac{P_{M+2} - P_M}{2M+3}\,P_{M-1}\right]$$

where

$$P_M = P_M(\cos\theta_0)$$

$$M = \begin{cases} N & \text{if } N \text{ is odd} \\ N+1 & \text{if } N \text{ is even} \end{cases}$$

From (6.9) it follows that $\delta\Delta g\,(N)$ is decreasing towards zero for increasing N, because:

$$P_N \to 0 \qquad \text{when} \qquad N \to \infty$$

For low order expansions it is shown in Figure 6.1 that the "errors" according to (6.9) are considerable. For example, the maximum error for N = 16 is 83 mgal. Again, it should be emphasized that these values are the deviations of the external series from the true series at the sphere of computation (cf. section 4.2). When N approaches infinity the sums of these two series are identical for exterior points. Thus the external series is indeed convergent and is a correct representation all the way down to the surface of a homogeneous ellipsoid. The same holds for a level ellipsoid. Hence, if we anticipate the mean earth ellipsoid as a level ellipsoid, the only contribution to the downward continuation error of Δg to the earth is given by the topography. The error coefficients $\bar{A}_{nm}$ and $\bar{B}_{nm}$ are then computed from formula (6.5) or (6.6) with the modification (cf. section 4.2):

$$r = \begin{cases} r_e & \text{if} \quad r < r_e \\ r & \text{if} \quad r \geq r_e \end{cases}$$

$$r_e = R\sqrt{(1-e^2)/(1-e^2\sin^2\theta)}$$

Formula (6.6′) should be used if the spherical harmonic expansion that is valid on the entire sphere of computation is required.

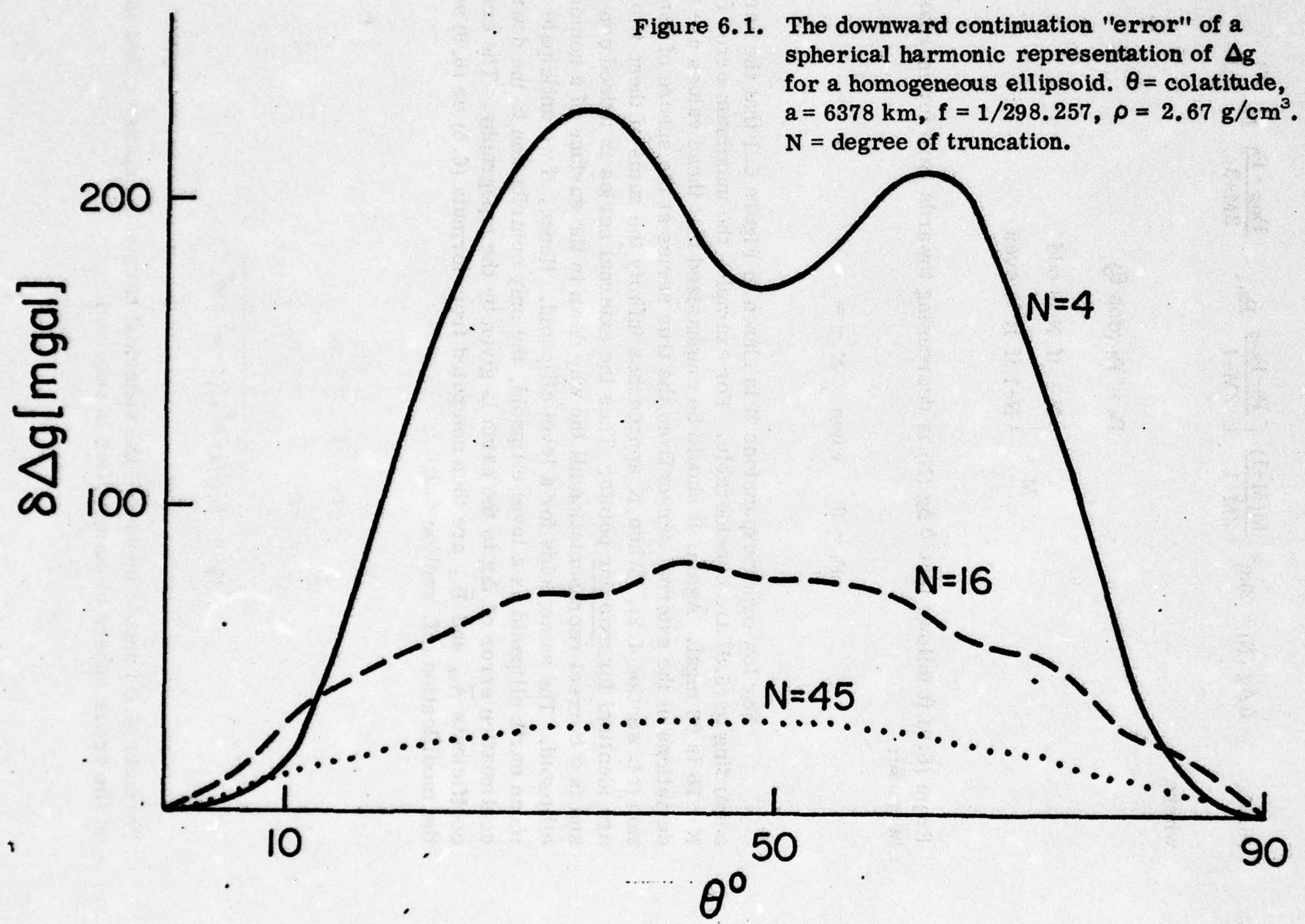

Figure 6.1. The downward continuation "error" of a spherical harmonic representation of Δg for a homogeneous ellipsoid. θ = colatitude, a = 6378 km, f = 1/298.257, ρ = 2.67 g/cm^3. N = degree of truncation.

6.2 Global RMS Errors

The total error of Δg for a truncated series of spherical harmonics is:

$$\epsilon_{\Delta g}(N) = \delta \Delta g(N) + e_{\Delta g}(N)$$

where $\delta \Delta g(N)$ is the error for the continuation and $e_{\Delta g}(N)$ is the truncation error. The global RMS of $\epsilon_{\Delta g}(N)$ is:

$$\|\epsilon_{\Delta g}(N)\| = [\,\|\delta \Delta g(N)\|^2 + \|e_{\Delta g}(N)\|^2\,]^{\frac{1}{2}} \tag{6.10}$$

At sea level we have:

$$\|e_{\Delta g}(N)\|^2 = \sum_{N+1}^{\infty} \sigma_n^2(\Delta g)$$

or, for

$$\sigma_n(\Delta g) = \frac{n-1}{r_o}\,\sigma_n$$

$$\|e_{\Delta g}(N)\|^2 = \sum_{N+1}^{\infty} \left(\frac{n-1}{r_o}\right)^2 \sigma_n^2 \tag{6.11}$$

where σ_n^2 are the disturbing potential degree variances.

From (6.1´) and (4.20) we finally obtain the approximate formula:

$$\|\delta \Delta g(N)\|^2 = \frac{1}{r_o^2}\sum_{n=0}^{N}\left[(n-1)\left(\frac{R}{r_o}\right)^{2n+1} + n+2\right]^2 p_n^2\,\sigma_n^2 \tag{6.12}$$

where

$$p_n = 1 - e^{-\alpha n}\,, \quad \alpha = 0.03$$

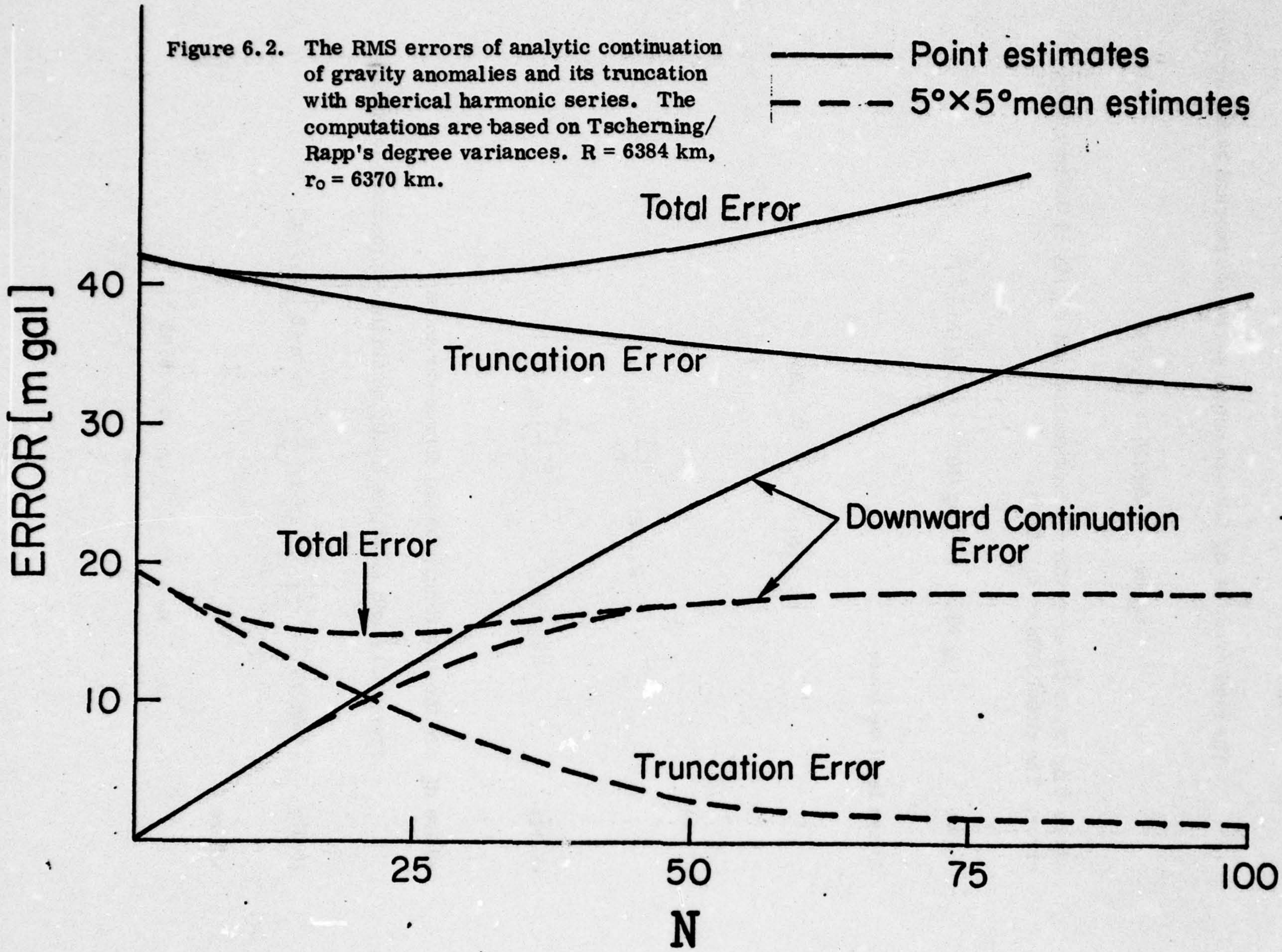

Figure 6.2. The RMS errors of analytic continuation of gravity anomalies and its truncation with spherical harmonic series. The computations are based on Tscherning/Rapp's degree variances. R = 6384 km, r_o = 6370 km.

The formulae (6.10) - (6.12) are shown in Figure 6.2 for the degree variances of Tscherning and Rapp (1974). In this figure we also depict the corresponding RMS errors for $5^\circ \times 5^\circ$ block mean anomalies, which are obtained by multiplying each degree variance of formulae (6.10) - (6.12) by the smoothing factor βn^2 (see section A.3 of the Appendix or Meissl, 1971).

7. The Error of Vertical Gradients of Gravity

The error of the vertical gradient of gravity is given by the derivative of $\delta\,\Delta g$ with respect to r:

$$\delta g_r = \delta \frac{\partial \Delta g}{\partial r} = \frac{\partial}{\partial r}\,\delta\,\Delta g = \sum_{n=0}^{\infty} \frac{\partial}{\partial r}\,\delta\,\Delta g_n \tag{7.1}$$

From formula (6.1) we obtain:

$$\frac{\partial}{\partial r}\,\delta\,\Delta g_n = -\frac{n-1}{r^2}\,\delta T_n + \frac{n-1}{r}\,\frac{\partial}{\partial r}\,\delta T_n - \frac{2n+1}{r^2}\,\Delta T_n + \frac{2n+1}{r}\,\frac{\partial}{\partial r}\,\Delta T_n \tag{7.2}$$

where ΔT_n is given in formula (4.17b). The derivative of ΔT_n with respect to r is (cf. Proposition 1, Appendix):

$$\frac{\partial}{\partial r}\,\Delta T_n = \frac{n}{r}\,\Delta T_n - r \iint_\sigma \mu\, P_n(\cos\psi)\, d\sigma$$

Using the approximation $\mu = \mu_0$ = constant, the last integral vanishes for $n \neq 0$. Thus we have:

$$\frac{\partial}{\partial r}\,\Delta T_n \approx \frac{n}{r}\,\Delta T_n\ ,\quad n \neq 0$$

Furthermore, we obtain from (5.2):

$$\frac{\partial}{\partial r}\,\delta T_n = -\frac{n+1}{r}\,\delta T_n - (2n+1)\,\frac{\Delta T_n}{r}$$

Inserting these derivatives into (7.2) we arrive at the following formula:

$$\delta \frac{\partial \Delta g_n}{\partial r} = - \frac{(n-1)(n+2)\delta T_n}{r^2} \tag{7.3}$$

This relation is the same as the relation between the spherical harmonics $\left(\frac{\partial \Delta g}{\partial r}\right)_n$ and T_n themselves in the external case. Thus we have:

$$\left| \frac{\delta \frac{\partial \Delta g_n}{\partial r}}{\frac{\partial \Delta g_n}{\partial r}} \right| = \left| \frac{\delta T_n}{T_n} \right| \tag{7.4}$$

The error of the vertical gradient of gravity can also be estimated from (7.1), (6.4), (6.5) and (6.7). From (6.4) we obtain:

$$\delta g_r = \sum_{n,m} [\overline{C}_{nm} \overline{R}_{nm} + \overline{D}_{nm} \overline{S}_{nm}]$$

where

$$\overline{C}_{nm} = \frac{\partial}{\partial r} \overline{A}_{nm}$$

$$\overline{D}_{nm} = \frac{\partial}{\partial r} \overline{B}_{nm}$$

Using the approximations (6.5) and (6.7) we arrive at:

$$\frac{\partial}{\partial r} (2n+1) \left[H + \frac{(n-1)(n+2)}{2 \times 3} \frac{H^3}{r^2} + \dots \right] = (2n+1) \left[-1 - \frac{(n-1)(n+2)}{2} \left(\frac{H}{r}\right)^2 \dots \right]$$

so that [for $(n,m) \neq (0,0)$]:

$$\left\{ \begin{matrix} \overline{C}_{nm} \\ \overline{D}_{nm} \end{matrix} \right\} = - \frac{G\rho_0 (n-1)(n+2)}{2 r^2} \iint H^2 \left\{ \begin{matrix} \overline{R}_{nm} \\ S_{nm} \end{matrix} \right\} d\sigma$$

Comparing this formula with (4.11) we finally obtain:

$$\begin{Bmatrix} \bar{C}_{nm} \\ \bar{D}_{nm} \end{Bmatrix} = - \frac{(n-1)(n+2)}{r^2} \begin{Bmatrix} \bar{a}_{nm} \\ \bar{b}_{nm} \end{Bmatrix}$$

This formula is equivalent with (7.3).

8. Computations with $5^\circ \times 5^\circ$ Mean Elevations

1654 mean elevation blocks were used to compute the downward continuation errors in a spherical harmonic expansion of the disturbing potential and the gravity anomaly. In all computations, the density of mass (ρ_0) was set to 2.67 g/cm^3. The spherical harmonic series were expanded to degree 16 (if not specified).

8.1 Computations for a Spherical Mean Earth

In the first set of computations we assumed a spherical mean earth with radius r = 6371 km. The downward continuation error for each harmonic of the disturbing potential and gravity anomaly can then be written according to formulae (4.11) and (6.6) respectively:

$$\begin{Bmatrix} \bar{a}_{nm} \\ \bar{b}_{nm} \end{Bmatrix} = \frac{G\rho_0}{2} \iint_\sigma H^2 \begin{Bmatrix} \bar{R}_{nm} \\ \bar{S}_{nm} \end{Bmatrix} d\sigma \tag{8.1}$$

and

$$\begin{Bmatrix} \bar{A}_{nm} \\ \bar{B}_{nm} \end{Bmatrix} = G\rho_0 \iint_\sigma H \begin{Bmatrix} \bar{R}_{nm} \\ \bar{S}_{nm} \end{Bmatrix} d\sigma \tag{8.2}$$

where

$$H = \begin{cases} r_s - r & \text{if } r_s > r \\ 0 & \text{otherwise} \end{cases}$$

The computed error coefficients are shown in Tables A.1 - A.2. From these coefficients the errors of V and Δg were computed from:

$$\delta V = \sum_{n=0}^{16} \sum_{m=0}^{n} [\bar{a}_{nm} \bar{R}_{nm}(\theta,\lambda) + \bar{b}_{nm} \bar{S}_{nm}(\theta,\lambda)] \tag{8.3}$$

and

$$(8.4) \qquad \delta\Delta g = \sum_{n=0}^{16} \sum_{m=0}^{n} [\bar{A}_{nm}\, \bar{R}_{nm}(\theta,\lambda) + \bar{B}_{nm}\, \bar{S}_{nm}(\theta,\lambda)]$$

The only area with any significant errors $\delta V/\gamma$ was south of the Himalayas. In order to estimate the errors at the surface of the earth in this area, a second computation was performed with $r = 6374$ km. Then the errors were interpolated between the computations with $r = 6371$ km and $r = 6374$ km. The maximum error estimate did not exceed half a meter.

The errors of the gravity anomalies were in the same order as the anomalies themselves (Fig. 8.1). These surprisingly large errors are not in agreement with the empirical results obtained through a direct comparison of satellite derived spherical harmonic expansions with terrestrial gravity anomalies. We conclude that the approximation of the mean earth with a sphere is too rough for estimating the anomaly errors.

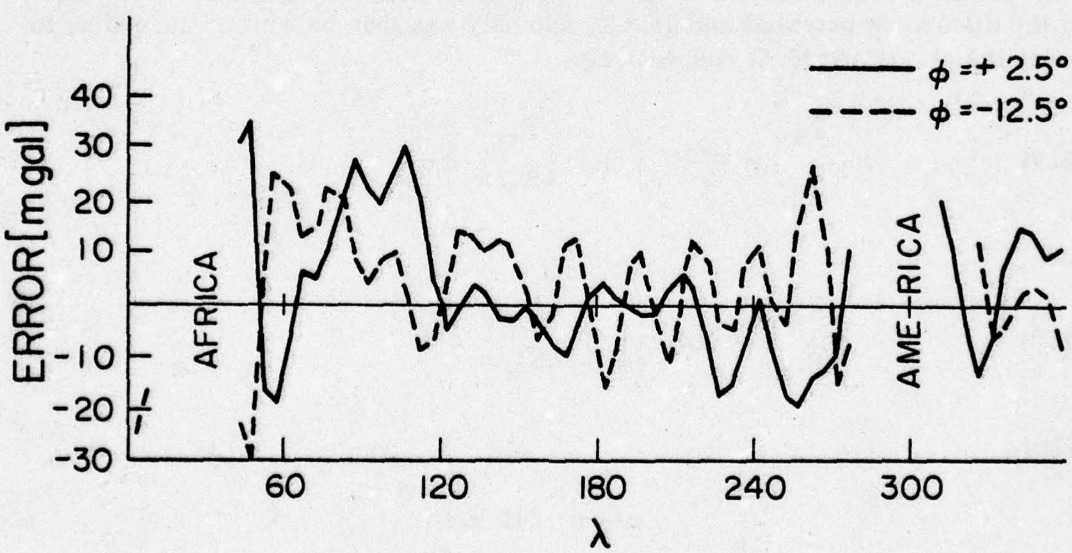

Figure 8.1. The downward continuation error of Δg along two profiles at sea level using formulae (8.2) and (8.4). Degree of truncation: 16.

8.2 Computations for an Ellipsoidal Mean Earth

We anticipated an ellipsoidal mean earth with the following parameters:

$$a = 6378140 \text{ km}$$

$$f = 1/298.257$$

The integration of the error coefficients for δV and $\delta \Delta g$ was performed according to formulae (4.11a) and (6.6´):

$$\left\{\begin{matrix}\bar{a}_{nm}\\ \bar{b}_{nm}\end{matrix}\right\} = \frac{G}{2} \iint [(\rho_0 - \rho_w)\, H_b^2 + \rho_w\, H^2] \left\{\begin{matrix}\bar{R}_{nm}\\ \bar{S}_{nm}\end{matrix}\right\} d\sigma \tag{8.5}$$

and

$$\left\{\begin{matrix}\bar{A}_{nm}\\ \bar{B}_{nm}\end{matrix}\right\} = G \iint [(\rho_0 - \rho)\, H_b + \rho_w\, H] \left\{\begin{matrix}\bar{R}_{nm}\\ \bar{S}_{nm}\end{matrix}\right\} d\sigma \tag{8.6}$$

where $\rho_0 = 2.67 \text{ g/cm}^3$ and $\rho_w = 1.03 \text{ g/cm}^3$.

The computed coefficients $\bar{a}_{00}$, $\bar{a}_{20}$, $\bar{a}_{40}$, $\bar{A}_{00}$, and $\bar{A}_{20}$ agreed very well with those determined in sections 4.2 and 6.1. The downward continuation "errors" for two profiles are shown in Figures 8.2 and 8.3. It was found that the geoidal undulation errors were small (less than a meter) while the gravity anomalies deviated up to 70 mgal in an expansion to degree 16 (cf. sections 4.2 and 6.1). The results are not in agreement with the empirical knowledge of the errors of downward continuation.

8.3 Computations for a Level Ellipsoid with Topography

In this section we make use of a mean earth level ellipsoid with the same dimensions as the ellipsoid of the previous section. For a level ellipsoid there is no contribution to the downward continuation error to the surface. We could therefore use the formulae (8.1) - (8.4) with the modification:

$$r = \begin{cases} r_s & \text{if } r_s > r \\ r & \text{if } r_s \leq r \end{cases} \tag{8.7}$$

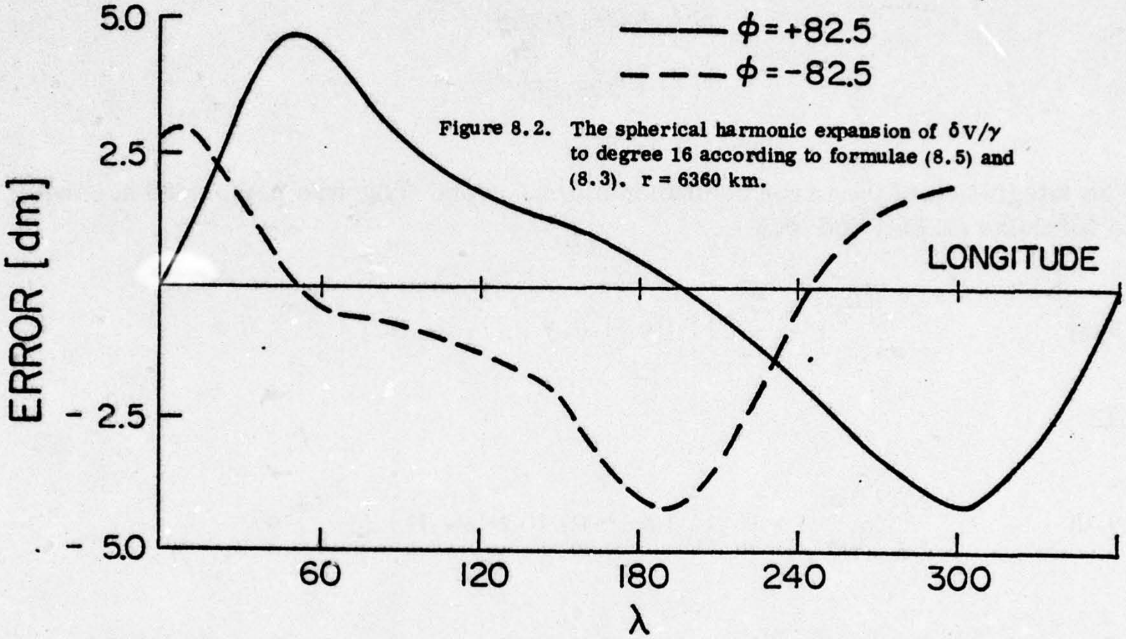

Figure 8.2. The spherical harmonic expansion of $\delta V/\gamma$ to degree 16 according to formulae (8.5) and (8.3). r = 6360 km.

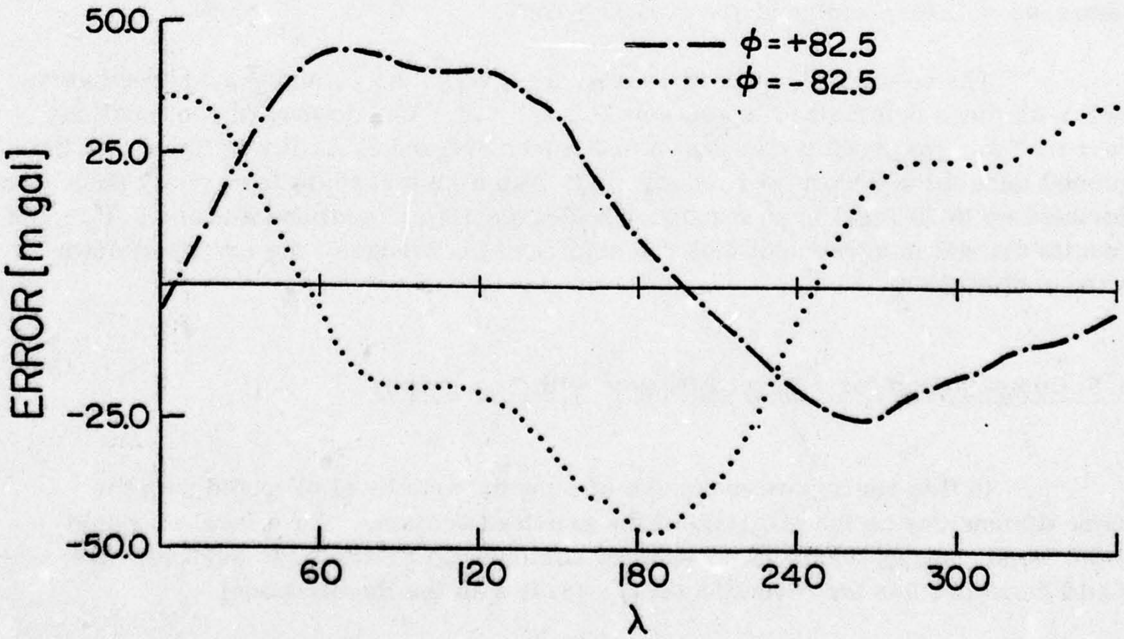

Figure 8.3. The spherical harmonic expansion of $\delta \Delta g$ according to formulae (8.6) and (8.4). N = 16, r = 6360 km.

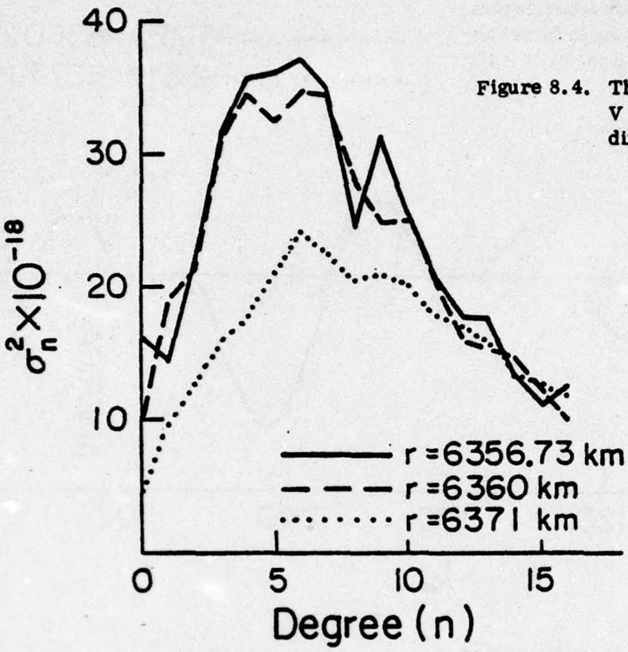

Figure 8.4. The degree variances of V [formula (8.8)] for different radii.

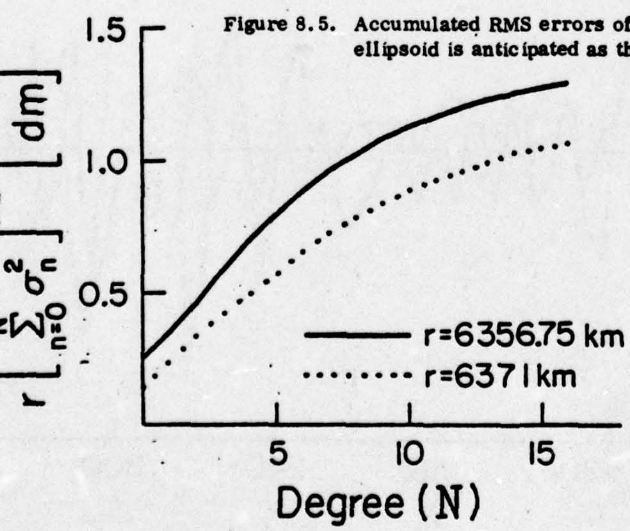

Figure 8.5. Accumulated RMS errors of $\delta V/\gamma$. A level ellipsoid is anticipated as the mean earth ellipsoid.

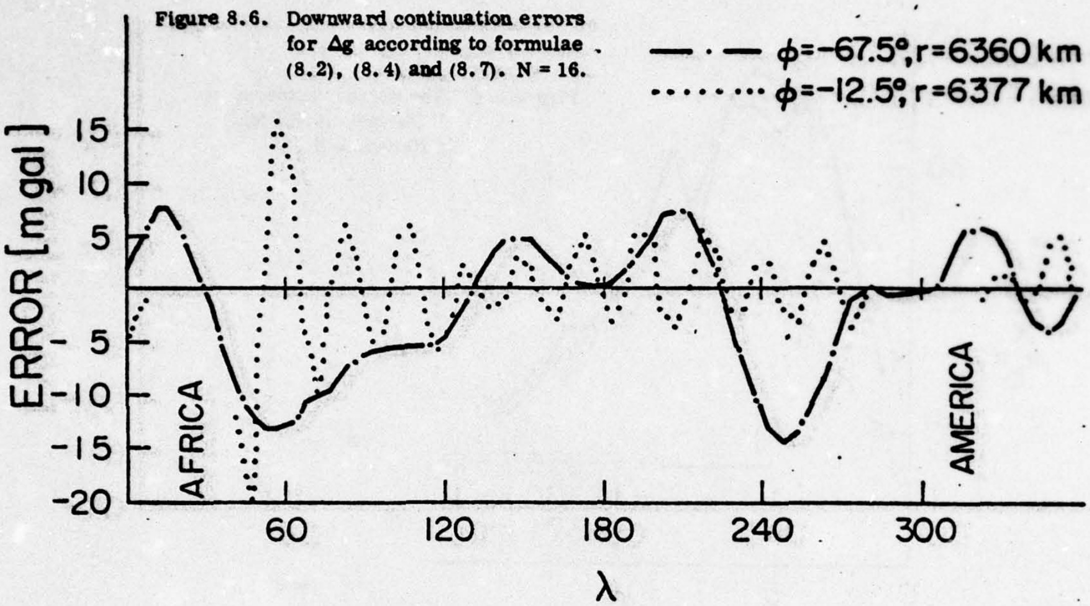

Figure 8.6. Downward continuation errors for Δg according to formulae (8.2), (8.4) and (8.7). N = 16.

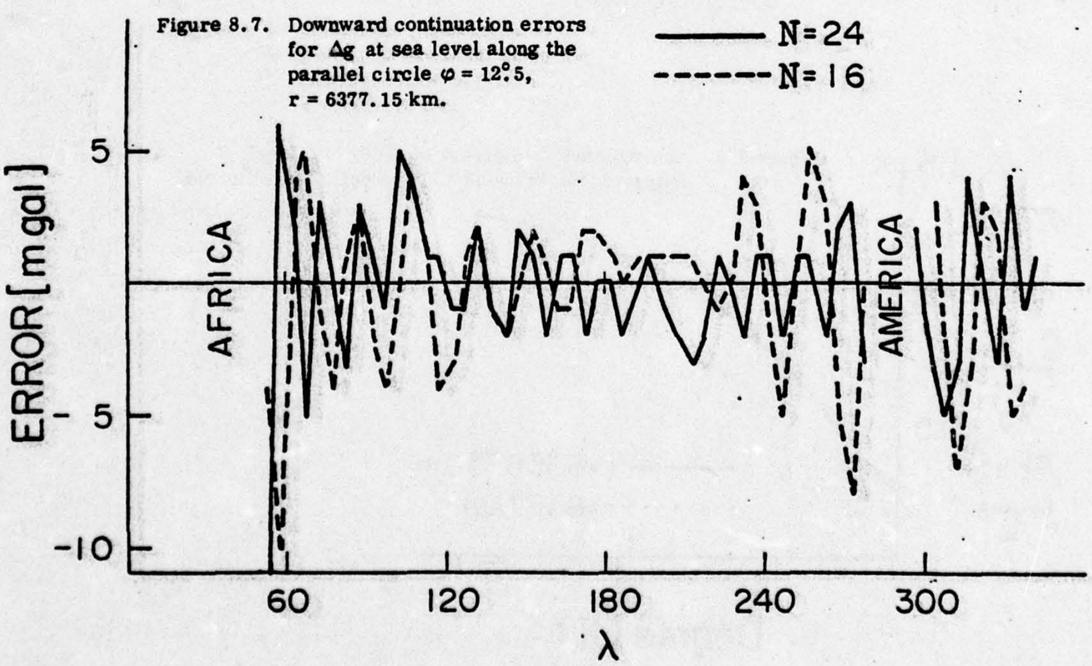

Figure 8.7. Downward continuation errors for Δg at sea level along the parallel circle $\varphi = 12°.5$, r = 6377.15 km.

where

$$r_s = a\sqrt{(1-e^2)/(1-e^2\sin^2\theta)}$$

For $r = a\sqrt{1-e^2}$ (at the poles), these computations are identical with those given in section 8.1. In Tables A.3 - A.4, we give the error coefficients for r = 6371 km. In Figure 8.4, we show the degree variances (σ_n^2) of the potential coefficients as defined by:

$$\sigma_n^2 = \left(\frac{r}{GM}\right)^2 \sum_{m=0}^{n} (\bar{a}_{nm}^{\,2} + \bar{b}_{nm}^{\,2}) \tag{8.8}$$

In Figure 8.5 the accumulated RMS errors of the geoidal undulations to degree $N_{max} = 16$ are shown. These errors do not exceed 0.13 m.

The downward continuation errors of the gravity anomalies in an expansion to degree 16 are depicted in Figure 8.6 along two latitudes at sea level. In Figure 8.7, the expansions to N = 16 and N = 24 are compared for a profile. Large error estimates are obtained at the edges of the continents. Otherwise, the errors are within ± 5 mgal.

9. Conclusions and Final Remarks

In this report we have investigated the downward continuation errors for spherical harmonic expansions of the gravity field of the earth. The simple model in section 3 showed that these errors are increasing with the height of the surrounding topography above the computation point. Furthermore, the relative errors are increasing with the degree of truncation of the series. It is also shown that the usual relation between the spherical harmonics of gravity anomalies and disturbing potentials is not valid for the errors of downward continuation:

$$\delta\Delta g_n \neq \frac{n-1}{r}\,\delta T_n \tag{9.1}$$

The errors of Δg_n are usually more serious than would be the case if this relation were true. One reason for this is that while the potential is dependent only upon the distance to the generating masses, the influence on the gravity anomalies from masses located above the computation point usually has opposite sign to that obtained in the downward continuation procedure. Thus, the relative errors of the gravity spherical harmonics are more than 100% in the model of section 3, where the only contribution to the anomaly is the disturbing mass outside the main sphere.

For the error of the vertical gradient of gravity (δg_r), the following formula was found for the model:

$$(\delta g_r)_n = -\frac{(n-1)(n+2)}{r^2}\,\delta T_n \qquad (9.2)$$

which is the same as the relation between the harmonics themselves.

Some general global RMS errors were estimated based on Tscherning/Rapp's degree variances. In the low order expansions ($N = 30$) the errors of geoidal undulations are 2.2 dm in the most optimistic estimate ($\alpha = 0.03$) while the RMS errors of gravity anomalies reach 16 mgal. These error estimates of the gravity anomalies seem too large, which might be due to the uncertainty in the basic approximation (4.17a).

Formulae were developed for a numerical integration of the downward continuation errors [formulae (8.1) through (8.7)]. Formulae (9.1) and (9.2) were found to be valid also for the real earth.

In sections 4.2 and 6.1 formulae (8.1) and (8.2) were tested at the surface of a homogeneous ellipsoid. The calculations showed that the "error" coefficients of degree 0, 2 and 4 were considerable. However, the errors $\delta V(N)$ and $\delta \Delta g(N)$ were attenuating with N to zero for N approaching infinity. The reason for this strange result is the following. Inside the minimum sphere, the potential of a homogeneous ellipsoid may be represented by two different spherical harmonic series: one which is convergent on the entire sphere of computation (also inside the ellipsoid) and one which is valid only outside the surface of matter. The latter is identical with the expansion outside the bounding sphere. As we are only interested in the errors on and outside the surface, we are looking for the downward continuation error of the latter development, while formulae (8.1) and (8.2) give the errors relative to the first series.

It was therefore suggested that formulae (8.1) and (8.2) should be modified (when applied to the real earth) in such a way that there are no contributions to the downward extension errors from the masses of the mean earth ellipsoid (level ellipsoid). This modification [formula (8.7)] implies that the errors of the spherical harmonic expansions are merely due to the topographical masses above the sphere of computation.

In the final computations with 1654 $5^\circ \times 5^\circ$ mean elevations (section 8.3) we arrived at very small errors of the geoidal undulations (RMS error = 0.13 m for $N = 16$). The degree variances of V were found to have a maximum 38×10^{-18} for $n = 6$ and $r = 6356.73$ km (at the poles).

Finally, we conclude that the gravity anomaly errors seem generally to be within ±5 mgal. However, at the edges of the continents and within rough areas on the continents, larger errors might be expected. Such areas may be studied in detail by utilizing the method described in section 8.3.

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Appendix A.1

A.1 Computation of "Minimum Sphere" Radius

The radius of the "minimum sphere" is computed as the maximum sum of the radius at sea level $b/\sqrt{1-e^2\cos^2\varphi}$, the height of the topography (h) and the geoidal undulation N:

$$R = \text{maximum earth } (R_h)$$

where

$$R_h = b/\sqrt{1-e^2\cos^2\varphi} + h + N$$

$$b = a\sqrt{1-e^2}$$

$$a = 6378.140 \text{ km}$$

$$e^2 = 6.694407 \times 10^{-3} \quad (f = 1/298.257)$$

The maximum was obtained for Mount Chimborazo in South America ($\varphi = -1.45°$), with:

$$R_h = 6378.126 + 6.267 + 0.010 = 6384.403 \text{ km}$$

Appendix A.2

A.2 Derivation of Some Formulae

Proposition 1:

(A.1)
$$\Delta_n = -\frac{\partial}{\partial r}\,\delta V_n$$

where

$$\delta V_n = \frac{1}{r}\iint_\sigma \int_r^{r_s} \mu\left[\left(\frac{r_1}{r}\right)^n - \left(\frac{r}{r_1}\right)^{n+1}\right] P_n(\cos\psi)\,d\nu$$

$$d\nu = r_1^2\,dr_1\,d\sigma$$

implies

$$\Delta_n = \frac{1}{r^2}\iint_\sigma \int_r^{r_s} \mu\left[(n+1)\left(\frac{r_1}{r}\right)^n + n\left(\frac{r}{r_1}\right)^{n+1}\right] P_n(\cos\psi)\,d\nu$$

Proof: We rewrite δV_n in the following way:

(A.2)
$$\delta V_n = \iint_\sigma K(r, r_s)\; P_n(\cos\psi)\,d\sigma$$

where

(A.3)
$$K(r, r_s) = \int_r^{r_s} k(r, r_1)\,dr_1$$

and

$$k(r, r_1) = \frac{\mu r_1^2}{r}\left[\left(\frac{r_1}{r}\right)^n - \left(\frac{r}{r_1}\right)^{n+1}\right]$$

Thus we obtain:

(A.4)
$$\frac{\partial}{\partial r} K(r, r_s) = \int_r^{r_s} \frac{\partial}{\partial r} k(r, r_1)\,dr_1 - k(r, r)$$

where

$$k(r, r) = 0 \tag{A.5}$$

and

$$\frac{\partial}{\partial r} k(r, r_1) = -\frac{\mu r_1^2}{r^2}\left[(n+1)\left(\frac{r_1}{r}\right)^n + n\left(\frac{r}{r_1}\right)^{n+1}\right] \tag{A.6}$$

By combining (A.1) through (A.6) we arrive at the proposition.

<u>Proposition 2:</u>

Given:

$$K(r, r_s) = r \begin{cases} 0 & \text{if } r \geq r_s \\ (n-1)\dfrac{(r_s/r)^{n+3}-1}{n+3} - (n+2)\dfrac{(r_s/r)^{-(n-2)}-1}{n-2} & \text{, if } r < r_s \\ & \quad n \neq 2 \\ \dfrac{(r_s/r)^5-1}{5} + 4 \ln(r_s/r) & \text{, if } r < r_s \text{ , } n = 2 \end{cases}$$

$K(r, r_s)$ can be expanded into the following series:

$$K(r, r_s) = (2n+1) H\left[1 + \frac{(n-1)(n+2)}{2 \times 3}\left(\frac{H}{r}\right)^2 + \ldots\right] \tag{A.7}$$

where

$$H = \begin{cases} 0 & \text{if} \quad r \geq r_s \\ r_s - r & \text{otherwise} \end{cases}$$

<u>Proof:</u> $n \neq 2$:

$$\left(\frac{r_s}{r}\right)^{n+3} = 1 + (n+3)\frac{H}{r} + \frac{(n+3)(n+2)}{2}\left(\frac{H}{r}\right)^2 + \frac{(n+3)(n+2)(n+1)}{2 \times 3}\left(\frac{H}{r}\right)^3 + \ldots$$

$$\left(\frac{r_s}{r}\right)^{-(n-2)} = 1 - (n-2)\frac{H}{r} + \frac{(n-2)(n-1)}{2}\left(\frac{H}{r}\right)^2 - \frac{(n-2)(n-1)n}{2 \times 3}\left(\frac{H}{r}\right)^3 + \ldots$$

Hence

$$K(r, r_s) = r\left[(n-1)\left\{\frac{H}{r} + \frac{n+2}{2}\left(\frac{H}{r}\right)^2 + \frac{(n+2)(n+1)}{2\times 3}\left(\frac{H}{r}\right)^3 + \dots\right\}\right.$$

$$\left.-(n+2)\quad\left\{-\frac{H}{r} + \frac{n-1}{2}\left(\frac{H}{r}\right)^2 - \frac{(n-1)n}{2\times 3}\left(\frac{H}{r}\right)^3 + \dots\right\}\right] =$$

$$(2n+1)\quad H\left[1 + \frac{(n-1)(n+2)}{2\times 3}\left(\frac{H}{r}\right)^2 - \dots\right]$$

n = 2:

$$\left(\frac{r_s}{r}\right)^5 = 1 + 5\,\frac{H}{r} + \frac{5x4}{2}\left(\frac{H}{r}\right)^2 + \frac{5x4x3}{2x3}\left(\frac{H}{r}\right)^3 + \dots$$

$$\ell n\,(r_s/r) = \frac{H}{r} - \frac{1}{2}\left(\frac{H}{r}\right)^2 + \frac{1}{3}\left(\frac{H}{r}\right)^3 - \dots$$

Hence

$$K(r, r_s) = r\left[\frac{H}{r} + 2\left(\frac{H}{r}\right)^2 + 2\left(\frac{H}{r}\right)^3 + \dots + 4\left\{\frac{H}{r} - \frac{1}{2}\left(\frac{H}{r}\right)^2 + \frac{1}{3}\left(\frac{H}{r}\right)^3 - \dots\right\}\right]$$

$$= 5\,H\left[1 + \frac{2}{3}\left(\frac{H}{r}\right)^2 + \dots\right]$$

This formula satisfies (A.7) for n = 2.

Proposition 3:

(A.8) $$\delta\Delta g(N) = \sum_{n=0}^{N} \overline{A}_n\,\overline{P}_n\,(t_0)$$

where

$$\text{(A.9)} \qquad \overline{A}_n = \int_{-t_o}^{t_o} (t_o^2 - t^2)\, \overline{P}_n(t)\, dt$$

implies

$$\delta\, \Delta g(N) = 2\, \frac{M(M+1)}{2M+1} \left[\frac{P_M - P_{M-2}}{2M-1}\, P_{M+1} - \frac{P_{M+2} - P_M}{2M+3}\, P_{M-1} \right]$$

where

$$P_M = P_M(t_o)$$

and

$$M = \begin{cases} N & \text{if } N \text{ is odd} \\ N+1 & \text{if } N \text{ is even} \end{cases}$$

<u>Proof:</u> We have:

$$\delta\, \Delta g(N) = \int_{-t_o}^{t_o} (t_o^2 - t^2) \sum_{n=0}^{N} \overline{P}_n(t)\, \overline{P}_n(t_o)\, dt$$

From Churchill (1963, p. 214), we obtain:

$$\text{(A.10)} \qquad \sum_{n=0}^{N} \overline{P}_n(t)\, \overline{P}_n(t_o) = \sum_{n=0}^{N} (2n+1)\, P_n(t)\, P_n(t_o) = (N+1)\, \frac{P_{N+1}(t) P_N(t_o) - P_{N+1}(t_o) P_N(t)}{t - t_o}$$

Thus we arrive at:

$$\delta\, \Delta g(N) = (N+1) \int_{-t_o}^{t_o} (t_o + t) \left[P_{N+1}(t_o) P_N(t) - P_{N+1}(t) P_N(t_o) \right] dt$$

$$= 2(N+1) \times \begin{cases} \displaystyle\int_0^{t_o} [t\, P_{N+1}(t_o) P_N(t) - t_o\, P_{N+1}(t) P_N(t_o)]\, dt\,, & N = \text{odd} \\ \displaystyle\int_0^{t_o} [t_o\, P_{N+1}(t_o)\, P_N(t) - t\, P_{N+1}(t) P_N(t_o)]\, dt\,, & N = \text{even} \end{cases}$$

In the last derivation we have used the fact that P_N is odd if N is odd and even if N is even.

We use the following substitution (see Churchill, 1963, p. 206):

$$\text{(A.11)} \qquad t\,P_N(t) = \frac{1}{2N+1}\left[(N+1)\,P_{N+1}(t) + N\,P_{N-1}(t)\right]$$

Then we obtain for odd N:

$$\text{(A.12)} \qquad \delta\Delta g(N) = 2\,\frac{(N+1)N}{2N+1}\int_0^{t_o}\left[P_{N+1}(t_o)\,P_{N-1}(t) - P_{N+1}(t)\,P_{N-1}(t_o)\right]dt$$

$$= 2\,\frac{(N+1)N}{2N+1}\left[\frac{P_N - P_{N-2}}{2N-1}\,P_{N+1} - \frac{P_{N+2} - P_N}{2N+3}\,P_{N-1}\right]$$

where we have used the abbreviation:

$$P_N = P_N(t_o)$$

(A.12) was obtained by the following relations from Churchill (1963, p. 207):

$$\text{(A.13)} \qquad (2n+1)\int P_n(t)\,dt = P_{n+1}(t) - P_{n-1}(t)$$

and

$$P_{2k+1}(0) = 0$$

For N even we obtain in the same way:

$$\delta\Delta g(N) = 2\,\frac{(N+1)(N+2)}{2N+3}\left[\frac{P_{N+1} - P_{N-1}}{2N+1}\,P_{N+2} - \frac{P_{N+3} - P_{N+1}}{2N+5}\,P_N\right]$$

<u>Proposition 4:</u>

$$\delta V(N) = \int_{-t_0}^{t_0} (t_0^2 - t^2)^2 \sum_{n=0}^{N} (2n+1)\, P_n(t)\, P_n(t_0)\, dt$$

implies

$$\delta V(N) = t_0^2 \; g_N(\bar{t})$$

where

$$g_N(t) = 2\,\frac{M(M+1)}{2M+1}\left[\frac{P_M(t) - P_{M-2}(t)}{2M-1}\, P_{M+1}(t_0) - \frac{P_{M+2}(t) - P_M(t)}{2M+3}\, P_{M-1}(t_0)\right]$$

$$M = \begin{cases} N & \text{if } N \text{ is odd} \\ N+1 & \text{if } N \text{ is even} \end{cases}$$

and

$$0 \le \bar{t} \le t_0$$

<u>Proof:</u> From the derivation of Prop. 3, we obtain (for odd N):

(A.14) $$\delta V(N) = \int_0^{t_0} (t_0^2 - t^2)\, g_N'(t)\, dt$$

where

$$g_N' = 2\,\frac{N(N+1)}{2N+1}\,[P_{N+1}(t_0)\, P_{N-1}(t) - P_{N+1}(t)\, P_{N-1}(t_0)]$$

Integrating (A.14) by parts, we obtain:

$$\delta V(N) = [(t_0^2 - t^2)\, g_N]_0^{t_0} + 2\int_0^{t_0} t\, g_N(t)\, dt$$

The first term is zero because $g_N(0) = 0$ and for the second term we may use the mean value theorem of integral calculus (Bartle, 1964, p. 303). The result is:

$$\text{(A.15)} \qquad \delta V(N) = 2 \int_0^{t_0} t\, g_N(t)\, dt = g_N(\bar{t})\, t_0^2$$

where

$$0 \le \bar{t} \le t_0$$

In the same way the proposition is proved for even N.

<u>Corollary 4:</u> $t_0 = 1 \rightarrow \delta V(N) = 0$ for $N \geq 4$.

<u>Proof:</u> We may integrate the first part of (A.15) one more time by parts. Then we obtain:

$$\delta V(N) = 2\, t_0\, G_N(t_0) - 2 \int_0^{t_0} G_N(t)\, dt$$

where

$$G_N(t) = \int g_N(t)\, dt$$

From the expression for g_N in Proposition 4 and (A.13) it follows that:

$$g_N(1) = G_N(1) = \int_0^1 G_N(t)\, dt = 0 \qquad \text{for } N \geq 4$$

which implies

$$\delta V(N) = 0 \quad \text{for } t_0 = 1$$

Appendix A.3

A.3 Estimates of the Optimum Degree of Expansion from Surface Mean Anomalies

We assume that we have a complete coverage of mean gravity anomalies (Δg) over blocks of size $\theta° \times \theta°$ on the mean earth sphere. As we have a finite set of mean anomalies, this mean gravity field can always be expanded into a finite series of spherical harmonics. If the number of coefficients (k) equals the number of mean anomalies (ℓ) there is a unique set of coefficients A_{nm}, B_{nm} that satisfies all equations:

$$\Delta \bar{g}_i = \sum_{n=0}^{N} \sum_{m=0}^{n} \frac{1}{\sigma_i} \iint_{\sigma_i} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda)\, \bar{P}_{nm}(\sin \varphi)\, d\sigma \tag{A.16}$$

where

$i = 1, 2, \ldots, \ell$

N = degree of truncation

σ_i = the area of the i th block

These coefficients can therefore be determined by solving a regular matrix equation, and the solution will give the exact mean value over each block.

Theoretically we are not limited to $\ell = k$ coefficients in order to obtain a solution that fits all $\Delta \bar{g}_i$. If $k > \ell$ (more coefficients than observations) the solution is not unique. One solution is thus obtained by using condition adjustment:

$$YA = \Delta \bar{g} \rightarrow A = Y^T (YY^T)^{-1} \Delta \bar{g}$$

where

A = vector of unknown coefficients

Y = vector of spherical harmonics (integrated over each block)

$\Delta \bar{g}$ = vector of mean anomalies

Thus there are different sets of coefficients that satisfy the observations and there is no evidence that one set is superior to another.

In a different approach we expand $\Delta\bar{g}$ into a complete set of spherical harmonics:

$$\Delta\bar{g} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (a_{nm} \cos m\lambda + b_{nm} \sin m\lambda)\, \bar{P}_{nm}(\sin\varphi) \tag{A.17}$$

where

$$\begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix} = \frac{1}{4\pi} \iint \Delta\bar{g} \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} \bar{P}_{nm}(\sin\varphi)\, d\sigma$$

If this series is truncated it will not be complete and will not satisfy the observed mean anomalies. This tendency has been verified by Rapp (1977, p. 42). Thus the commonly used rule of the optimum degree of truncation:

$$N_{\text{optimum}} = \frac{180^\circ}{\theta^\circ} \tag{A.18}$$

where θ° is the block size, does not hold for mean anomalies.

We may then ask whether (A.18) is valid for estimating point anomalies (Δg). We assume that Δg may be expanded into a convergent series at each point of the mean earth sphere:

$$\Delta g = \sum_{n=0}^{\infty} \Delta g_n \tag{A.19}$$

where

$$\Delta g_n = \sum_{m=0}^{n} (\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda)\, \bar{P}_{nm}(\sin\varphi)$$

The smoothed field (A.17) is then related to Δg according to:

$$\Delta\bar{g}(x) = \iint_\sigma B(x\cdot y)\, \Delta g(y)\, d\sigma_y = \sum_{n=0}^{\infty} \Delta g_n(x)\, \beta_n \tag{A.20}$$

where $B(x\cdot y)$ is the integral kernel of the smoothing operator and β_n are its eigen values.

Assuming that each block has the same size as a circular disk of spherical radius:

$$\psi_0 = (\theta \sin \theta / \pi)^{\frac{1}{2}}$$

we obtain from Meissl (1971):

$$B(x \cdot y) = \begin{cases} \dfrac{1}{2\pi} \dfrac{1}{1-\cos\psi_0} & x \cdot y \geq \cos \psi_0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta_n = \frac{1}{1-\cos\psi_0} \quad \frac{P_{n-1}(\cos\psi_0) - P_{n+1}(\cos\psi_0)}{2n+1}$$

Comparing (A.17), (A.19) and (A.20) we obtain:

$$\text{(A.21)} \qquad \begin{Bmatrix} \overline{C}_{nm} \\ \overline{S}_{nm} \end{Bmatrix} = \frac{1}{\beta_n} \begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix}$$

Thus the correct spherical harmonic coefficients of Δg are given by the coefficients of the mean anomalies divided by β_n. Theoretically this estimate of Δg is improved for each additional coefficient included in the series (A.19). However, the estimates of the coefficients according to (A.21) are poor for higher degrees, because β_n approaches zero. Thus we have in practice to limit ourselves to a finite number of coefficients.

Let us now assume that we estimate Δg by a truncated form of (A.20):

$$\hat{\Delta g} = \sum_{n=0}^{N} \Delta g_n \, \beta_n$$

The error of this estimate is then given by:

$$\hat{\Delta g} - \Delta g = \sum_{n=0}^{N} \Delta g_n (\beta_n - 1) - \sum_{n=N+1}^{\infty} \Delta g_n$$

As the spherical harmonics of different degrees are orthogonal to each other, we obtain the following global RMS error:

$$\| \hat{\Delta g} - \Delta g \| = \left[\sum_{n=0}^{N} \sigma_n^2 (1 - \beta_n)^2 + \sum_{n=N+1}^{\infty} \sigma_n^2 \right]^{\frac{1}{2}}$$

or

$$\| \hat{\Delta g} - \Delta g \| = \left[\sum_{n=0}^{\infty} \sigma_n^2 - \sum_{n=0}^{N} \sigma_n^2 \, \beta_n (2 - \beta_n) \right]^{\frac{1}{2}} \tag{A.22}$$

where

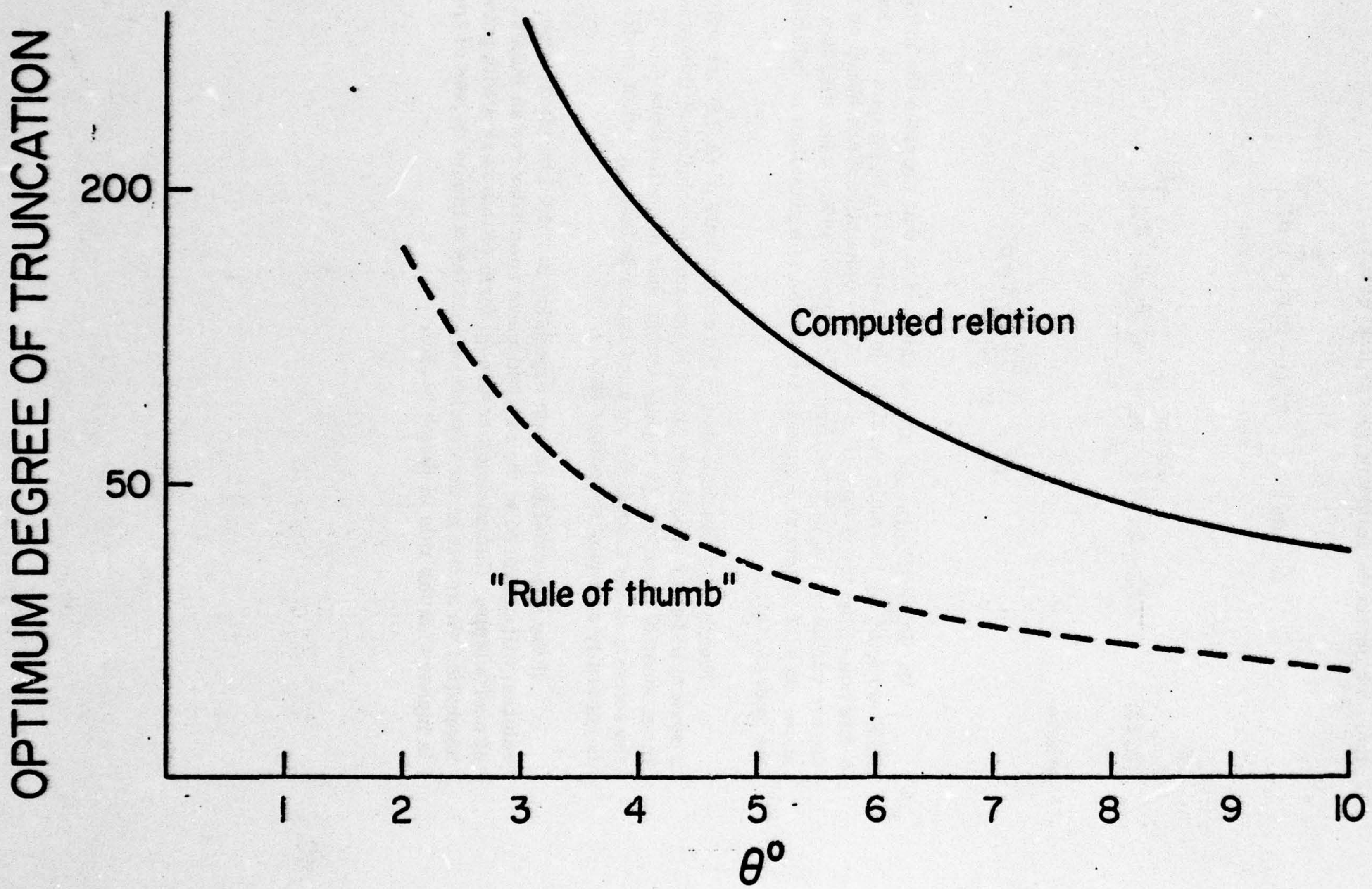
$$\sigma_n^2 = \frac{1}{4\pi} \iint \Delta g_n^2 \, d\sigma$$

We define $N_{optimum}$ as the value of N that provides the minimum RMS error of $\hat{\Delta g}$ [formula (A.22)]. In Figure A.1, $N_{optimum}$ is given as a function of the block size $\theta° \times \theta°$. The computations are based on the degree variances (σ_n^2) of Tscherning and Rapp (1974). The diagram clearly shows that the optimum degree of truncation is higher than is indicated by the rule (A.18).

Finally, we conclude that if the coefficients of (A.16) are solved from a system of linear equations, there is one unique solution of this system if the number of equations (ℓ) equals the number of coefficients (k). If $k > \ell$ the solution is not unique. We may not conclude that one set of coefficients is generally superior to another set.

If the spherical harmonic expansion is solved by integration, the solution will converge to the original mean anomalies for an infinite number of coefficients. Comparing a truncated form of this series with point anomalies we arrive at the conclusion that the optimum degree of truncation is higher than the rule of thumb (A.18).

Figure A.1. The optimum degree of truncation of a spherical harmonic expansion of the gravity field from surface mean anomalies of block size $\theta° \times \theta°$. The computations are based on the degree variances of Tscherning and Rapp (1974).



N	M	A	B	N	M	A	B
0		4.01					
1		2.30					
1	1	0.86	2.94				
2		0.51					
2	1	-0.36	3.64	2	2	-2.81	0.70
3		-2.81					
3	1	-0.50	1.67	3	2	-4.35	0.82
3	3	-0.09	-1.13				
4		-1.80					
4	1	-0.36	-2.15	4	2	-3.53	0.23
4	3	0.58	-3.52	4	4	1.41	0.76
5		-1.84					
5	1	0.26	-2.61	5	2	-0.35	-0.31
5	3	0.48	-3.24	5	4	3.22	-1.10
5	5	-0.36	1.83				
6		2.21					
6	1	-0.44	-2.26	6	2	2.82	-0.07
6	3	0.30	-0.79	6	4	3.41	-0.75
6	5	-0.60	2.12	6	6	-1.15	-0.00
7		1.84					
7	1	0.69	0.94	7	2	3.16	-0.56
7	3	0.62	1.51	7	4	1.62	-0.13
7	5	-0.11	3.00	7	6	-1.98	0.21
7	7	-0.32	-1.22				
8		1.10					
8	1	-0.03	1.90	8	2	0.98	-0.32
8	3	0.30	2.36	8	4	-0.82	-0.21
8	5	-0.30	2.29	8	6	-1.95	0.41
8	7	0.72	-1.48	8	8	-0.00	-0.32
9		-1.18					
9	1	0.37	2.59	9	2	-1.50	-0.08
9	3	0.00	1.49	9	4	-2.46	0.43
9	5	-0.66	-0.01	9	6	-1.65	0.50
9	7	0.39	-2.43	9	8	1.72	0.09
9	9	-0.18	0.35				
10		-1.63					
10	1	-0.10	-0.82	10	2	-2.39	0.69
10	3	-0.20	-0.97	10	4	-2.06	0.95
10	5	-0.27	-1.49	10	6	-0.50	0.23
10	7	0.19	-2.19	10	8	1.15	0.04
10	9	0.36	0.89	10	10	0.11	-0.08
11		-1.19					
11	1	0.28	-1.68	11	2	-1.18	-0.06
11	3	-0.15	-2.01	11	4	-0.33	0.05
11	5	0.06	-1.54	11	6	1.25	-0.20
11	7	0.22	-1.00	11	8	1.15	-0.09
11	9	-0.44	1.94	11	10	-0.43	-0.05
11	11	0.41	-0.33				
12		0.40					
12	1	-0.24	-2.05	12	2	0.92	0.09
12	3	-0.63	-1.48	12	4	1.30	-0.25
12	5	-0.06	-0.77	12	6	1.80	-0.40
12	7	-0.04	0.17	12	8	0.80	-0.18
12	9	-0.13	1.69	12	10	-0.76	0.11
12	11	-0.20	-0.05	12	12	0.15	-0.09
13		1.84					
13	1	0.06	0.30	13	2	2.18	-0.46
13	3	0.07	0.34	13	4	1.34	-0.44
13	5	0.25	0.79	13	6	0.82	-0.39
13	7	-0.02	1.28	13	8	-0.67	-0.15
13	9	0.35	1.01	13	10	-0.80	-0.26
13	11	0.22	-1.01	13	12	0.03	0.75
13	13	-0.19	0.10				
14		0.65					
14	1	0.22	1.19	14	2	0.90	-0.34
14	3	0.21	1.29	14	4	0.43	-0.44
14	5	0.48	1.48	14	6	-0.33	-0.31
14	7	0.16	1.43	14	8	-1.27	0.20
14	9	0.11	0.38	14	10	-0.38	-0.22
14	11	-0.09	-1.08	14	12	0.30	-0.25
14	13	0.06	0.54	14	14	-0.19	0.17
15		-0.12					
15	1	-0.08	1.97	15	2	-0.52	-0.40
15	3	0.29	1.44	15	4	-1.06	-0.37
15	5	0.20	0.87	15	6	-1.10	-0.07
15	7	0.05	0.19	15	8	-0.84	0.44
15	9	0.10	-0.62	15	10	0.25	0.10
15	11	-0.18	-0.70	15	12	0.38	0.35
15	13	-0.27	-0.02	15	14	-0.04	-0.25
15	15	-0.04	0.13				
16		-1.28					
16	1	0.07	0.15	16	2	-1.73	0.32
16	3	-0.18	-0.16	16	4	-1.27	0.61
16	5	-0.09	-0.47	16	6	-1.13	0.39
16	7	-0.08	-0.90	16	8	-0.28	0.38
16	9	0.03	-1.12	16	10	0.65	-0.12
16	11	-0.04	-0.27	16	12	0.34	0.55
16	13	0.19	0.56	16	14	-0.25	-0.18
16	15	-0.29	-0.44	16	16	0.07	-0.08

Table A.1. Computed error coefficients of the spherical harmonic expansion of T at sea level. A spherical mean earth is anticipated. The coefficients are normalized:
$A = a_{nm} \times r \times 10^9/GM$, $B = b_{nm} \times r \times 10^9/GM$

N	M	A	B	N	M	A	B
0		37.21					
1		16.82					
1	1	12.87	15.67				
2		7.09					
2	1	-1.31	17.85	2	2	-12.06	7.27
3		-16.82					
3	1	-7.78	9.04	3	2	-22.13	9.83
3	3	1.05	4.69				
4		-4.23					
4	1	-6.01	-11.66	4	2	-22.70	1.37
4	3	7.12	-12.27	4	4	2.98	12.13
5		-17.55					
5	1	1.69	-8.63	5	2	-6.90	-4.52
5	3	6.05	-13.06	5	4	15.65	-7.99
5	5	-4.76	11.88				
6		11.28					
6	1	-3.28	-11.47	6	2	9.39	-1.60
6	3	3.53	-3.00	6	4	13.99	-6.09
6	5	-5.95	4.22	6	6	-5.38	1.34
7		2.85					
7	1	6.04	6.15	7	2	15.35	-3.75
7	3	4.74	4.24	7	4	4.73	-0.25
7	5	-2.40	8.18	7	6	-5.32	-1.25
7	7	-3.10	-6.47				
8		4.87					
8	1	0.33	1.88	8	2	7.05	-2.64
8	3	4.10	7.12	8	4	-2.27	-0.31
8	5	-4.60	8.66	8	6	-2.31	2.79
8	7	8.63	-3.83	8	8	-3.74	-3.46
9		-6.35					
9	1	3.76	10.52	9	2	-2.39	-1.00
9	3	2.19	4.85	9	4	-9.06	3.16
9	5	-4.63	0.53	9	6	-3.46	3.84
9	7	5.58	-7.54	9	8	7.68	2.11
9	9	-2.51	-0.44				
10		-4.60					
10	1	-0.80	-5.45	10	2	-8.65	3.45
10	3	-2.33	-4.73	10	4	-8.63	6.06
10	5	-1.51	-3.08	10	6	-1.07	-0.40
10	7	2.91	-6.17	10	8	1.89	0.28
10	9	3.59	3.84	10	10	2.59	-2.04
11		-4.64					
11	1	4.56	-4.00	11	2	-2.84	-1.44
11	3	-1.40	-6.37	11	4	-2.83	-0.21
11	5	0.47	-1.50	11	6	4.03	-0.12
11	7	0.88	-4.03	11	8	0.96	0.05
11	9	-3.57	7.50	11	10	2.47	2.16
11	11	3.14	-2.51				
12		-0.27					
12	1	-0.64	-7.77	12	2	1.49	-0.76
12	3	-4.44	-4.60	12	4	1.19	-2.00
12	5	0.22	-1.18	12	6	5.16	-0.73
12	7	-2.18	-1.25	12	8	-0.23	-0.11
12	9	-0.92	4.82	12	10	-0.71	0.85
12	11	-0.50	3.18	12	12	-0.45	-1.44
13		7.53					
13	1	1.69	3.58	13	2	7.10	-3.64
13	3	-0.60	0.64	13	4	0.01	-3.22
13	5	1.33	1.44	13	6	1.96	-2.23
13	7	-1.87	2.55	13	8	-3.58	-1.74
13	9	2.38	3.77	13	10	0.50	-2.08
13	11	1.32	-3.16	13	12	-2.20	4.98
13	13	-1.29	0.24				
14		-0.74					
14	1	1.11	2.12	14	2	0.83	-2.55
14	3	0.63	1.26	14	4	-0.01	-2.63
14	5	2.51	2.80	14	6	-0.22	-1.83
14	7	0.04	4.19	14	8	-4.02	0.80
14	9	1.19	1.81	14	10	2.40	-0.30
14	11	-0.93	-3.12	14	12	-0.89	-1.53
14	13	-0.56	2.39	14	14	-0.18	1.46
15		1.90					
15	1	-0.20	4.84	15	2	-1.03	-2.74
15	3	2.57	3.63	15	4	-3.07	-2.21
15	5	0.98	1.76	15	6	-1.40	-1.08
15	7	0.11	0.86	15	8	-1.77	2.72
15	9	1.78	-0.55	15	10	1.02	0.21
15	11	-0.86	-0.97	15	12	-1.54	2.18
15	13	-0.74	-2.12	15	14	-0.49	-1.85
15	15	0.21	0.63				
16		-4.14					
16	1	-1.17	0.29	16	2	-5.43	1.65
16	3	-2.25	-2.55	16	4	-2.01	4.07
16	5	-1.36	-1.28	16	6	-2.94	0.27
16	7	0.31	-2.13	16	8	-0.65	2.05
16	9	0.75	-2.21	16	10	1.93	-0.74
16	11	-1.36	-0.12	16	12	-0.25	2.78
16	13	1.75	2.10	16	14	1.41	-1.94
16	15	-3.05	-2.89	16	16	0.22	-1.44

Table A.2. Computed error coefficients of the spherical harmonic expansion of Δg at sea level. A spherical mean earth is anticipated. Unit: mgal.

N	M	A	B	N	M	A	B
0		2.12					
1		1.75					
1	1	0.53	2.51				
2		-0.56					
2	1	0.25	2.81	2	2	-1.97	0.62
3		-2.09					
3	1	-0.27	0.80	3	2	-3.01	0.45
3	3	-0.03	-1.21				
4		-1.58					
4	1	-0.26	-1.67	4	2	-2.06	-0.03
4	3	-0.04	-2.55	4	4	1.26	0.24
5		0.15					
5	1	0.07	-2.52	5	2	0.51	-0.20
5	3	-0.06	-2.50	5	4	2.53	-0.19
5	5	-0.37	1.25				
6		1.60					
6	1	0.12	-1.21	6	2	2.55	-0.11
6	3	-0.08	-0.48	6	4	2.65	-0.57
6	5	-0.12	2.13	6	6	-1.13	-0.07
7		1.69					
7	1	-0.03	1.07	7	2	2.25	-0.12
7	3	-0.04	1.83	7	4	1.04	-0.33
7	5	0.21	2.25	7	6	-1.73	0.08
7	7	-0.05	-0.91				
8		0.32					
8	1	0.00	2.36	8	2	0.01	-0.11
8	3	0.09	2.33	8	4	-1.18	0.17
8	5	0.19	1.33	8	6	-1.87	0.29
8	7	0.26	-1.52	8	8	0.45	-0.21
9		-1.35					
9	1	0.09	1.52	9	2	-2.04	0.11
9	3	0.17	0.82	9	4	-2.22	0.35
9	5	-0.01	-0.33	9	6	-1.42	0.30
9	7	0.37	-1.78	9	8	1.14	0.01
9	9	0.07	0.23				
10		-1.75					
10	1	0.01	-0.67	10	2	-2.18	0.28
10	3	0.06	-1.13	10	4	-1.40	0.24
10	5	-0.06	-1.66	10	6	-0.24	0.10
10	7	0.07	-1.54	10	8	1.56	0.16
10	9	-0.05	0.97	10	10	0.05	-0.07
11		-0.50					
11	1	-0.12	-2.14	11	2	-0.45	0.12
11	3	-0.14	-1.95	11	4	0.40	0.06
11	5	-0.05	-1.63	11	6	1.12	-0.11
11	7	-0.23	-0.62	11	8	1.34	-0.01
11	9	-0.22	1.55	11	10	-0.43	0.03
11	11	0.25	-0.09				
12		1.11					
12	1	-0.07	-1.58	12	2	1.54	-0.15
12	3	-0.20	-1.14	12	4	1.67	-0.16
12	5	-0.10	-0.32	12	6	1.57	-0.18
12	7	-0.22	0.67	12	8	0.59	-0.23
12	9	-0.18	1.44	12	10	-0.95	0.00
12	11	0.12	-0.37	12	12	0.12	0.21
13		1.56					
13	1	0.05	0.32	13	2	2.04	-0.24
13	3	-0.04	0.55	13	4	1.50	-0.32
13	5	-0.07	1.09	13	6	0.74	-0.16
13	7	-0.04	1.45	13	8	-0.35	-0.17
13	9	0.06	0.77	13	10	-1.01	-0.12
13	11	0.03	-0.81	13	12	0.14	0.20
13	13	-0.09	0.26				
14		0.61					
14	1	0.08	1.77	14	2	0.73	-0.11
14	3	0.16	1.68	14	4	0.13	-0.20
14	5	0.10	1.46	14	6	-0.57	-0.07
14	7	0.09	1.08	14	8	-1.03	0.05
14	9	0.21	-0.11	14	10	-0.61	-0.08
14	11	0.10	-1.06	14	12	0.32	0.09
14	13	-0.16	0.27	14	14	-0.15	0.09
15		-0.76					
15	1	0.05	1.55	15	2	-1.10	0.08
15	3	0.18	1.28	15	4	-1.22	0.14
15	5	0.17	0.65	15	6	-1.26	0.09
15	7	0.16	-0.08	15	8	-0.99	0.17
15	9	0.11	-0.84	15	10	-0.03	0.07
15	11	0.11	-0.81	15	12	0.58	0.09
15	13	-0.06	0.32	15	14	-0.17	-0.09
15	15	-0.17	-0.02				
16		-1.33					
16	1	-0.00	0.01	16	2	-1.78	0.20
16	3	0.03	-0.21	16	4	-1.43	0.32
16	5	0.05	-0.58	16	6	-0.91	0.21
16	7	0.12	-1.04	16	8	-0.20	0.16
16	9	-0.07	-1.01	16	10	0.47	0.06
16	11	-0.03	-0.20	16	12	0.55	0.15
16	13	0.07	0.50	16	14	-0.09	-0.20
16	15	-0.09	-0.17	16	16	-0.14	-0.10

Table A.3. Computed error coefficients of the spherical harmonic expansion of T for r = 6371 km. The mean earth is anticipated as a level ellipsoid with a = 6378.140 km and f = 298.56. The coefficients are normalized: $A = a_{nm} \times r \times 10^9/GM$, $B = b_{nm} \times r \times 10^9/GM$.

N	M	A	B	N	M	A	B
0		16.16					
1		10.75					
1	1	9.27	13.10				
2		-7.51					
2	1	4.32	13.09	2	2	-6.25	7.93
3		-13.88					
3	1	-4.97	1.82	3	2	-12.19	5.78
3	3	1.47	1.45				
4		-5.74					
4	1	-4.87	-8.86	4	2	-10.34	-0.77
4	3	1.69	-6.87	4	4	4.65	6.14
5		3.88					
5	1	1.18	-10.29	5	2	0.97	-3.26
5	3	0.29	-11.45	5	4	10.42	-0.37
5	5	-4.25	7.26				
6		7.99					
6	1	2.78	-3.41	6	2	11.45	-1.73
6	3	-1.36	-3.44	6	4	10.86	-6.52
6	5	-2.46	7.50	6	6	-5.45	-0.56
7		6.82					
7	1	0.41	5.42	7	2	10.59	-0.56
7	3	-1.34	7.47	7	4	3.36	-3.39
7	5	0.43	5.50	7	6	-4.74	0.28
7	7	-0.66	-4.62				
8		0.61					
8	1	-0.31	9.24	8	2	0.18	0.13
8	3	0.34	8.84	8	4	-5.63	2.42
8	5	0.56	3.50	8	6	-3.29	1.73
8	7	4.41	-5.28	8	8	-0.52	-1.94
9		-6.92					
9	1	0.14	4.81	9	2	-9.08	1.55
9	3	1.58	2.22	9	4	-8.23	3.12
9	5	-0.06	-0.31	9	6	-3.26	1.84
9	7	6.17	-4.57	9	8	3.89	0.12
9	9	-0.23	-0.29				
10		-7.98					
10	1	-0.63	-3.67	10	2	-9.35	1.87
10	3	0.77	-3.95	10	4	-3.91	1.19
10	5	0.61	-4.69	10	6	-1.37	0.37
10	7	1.94	-4.13	10	8	5.42	1.74
10	9	-0.17	3.57	10	10	2.74	-0.77
11		-1.05					
11	1	-1.08	-7.93	11	2	-1.62	0.16
11	3	-1.03	-5.89	11	4	2.11	0.35
11	5	0.60	-4.88	11	6	2.60	-0.79
11	7	-2.24	-2.57	11	8	2.59	0.43
11	9	-1.40	6.03	11	10	1.42	0.93
11	11	2.34	-0.28				
12		5.90					
12	1	0.06	-4.64	12	2	6.53	-1.43
12	3	-1.39	-3.68	12	4	5.32	-0.81
12	5	-0.74	-0.68	12	6	3.96	-0.75
12	7	-2.63	1.09	12	8	-0.69	-1.46
12	9	-1.81	4.49	12	10	-0.81	0.84
12	11	1.29	0.11	12	12	-0.43	0.74
13		6.13					
13	1	0.99	1.89	13	2	7.88	-1.45
13	3	0.04	1.30	13	4	4.30	-2.48
13	5	-0.89	3.00	13	6	1.42	-0.57
13	7	-1.33	4.22	13	8	-1.02	-1.33
13	9	0.02	1.71	13	10	-0.97	-0.91
13	11	0.37	-1.55	13	12	-1.62	1.46
13	13	-0.87	1.12				
14		1.30					
14	1	0.63	5.76	14	2	2.26	-0.58
14	3	1.15	5.07	14	4	0.26	-1.76
14	5	0.47	3.29	14	6	-1.28	-0.58
14	7	-0.00	3.63	14	8	-1.79	-0.03
14	9	1.76	-0.34	14	10	0.19	-0.91
14	11	0.72	-3.15	14	12	-1.35	0.86
14	13	-1.36	0.68	14	14	-0.51	0.75
15		-3.27					
15	1	-0.03	4.63	15	2	-4.03	0.62
15	3	0.70	3.98	15	4	-3.55	1.20
15	5	0.80	1.41	15	6	-2.08	0.11
15	7	1.38	0.12	15	8	-2.00	0.81
15	9	1.26	-2.31	15	10	0.53	0.42
15	11	0.72	-2.15	15	12	-0.01	0.78
15	13	-0.20	0.13	15	14	-0.09	-0.97
15	15	-1.22	-0.35				
16		-4.51					
16	1	-0.43	-0.09	16	2	-5.66	1.54
16	3	-0.24	-0.43	16	4	-3.99	2.64
16	5	-0.17	-0.77	16	6	-1.79	1.11
16	7	1.71	-2.68	16	8	-0.38	0.96
16	9	-0.31	-3.10	16	10	0.24	0.35
16	11	-0.35	-0.03	16	12	0.48	1.40
16	13	1.20	1.04	16	14	1.47	-1.88
16	15	-0.84	-1.15	16	16	-1.29	-1.40

Table A.4. Computed error coefficients of the spherical harmonic expansion of Δg for $r = 6371$ km. The mean earth is anticipated as a level ellipsoid with $a = 6378.140$ km and $f = 298.56$. Unit: mgal.